SHEAR ANISOTROPIC INHOMOGENEOUS BESOV AND TRIEBEL-LIZORKIN SPACES IN $\mathbb{R}^d$

DANIEL VERA

ABSTRACT. We define distribution spaces in $\mathbb{R}^d$ via $\ell^q(L^p)$ or $L^p(\ell^q)$ norms of a sequence of convolutions of $f \in \mathcal{S}'$ with smooth functions, the shearlet system. Then, we define associated sequence spaces and prove characterizations. We also show a reproducing identity in $\mathcal{S}'$. Finally, we prove Sobolev-type embeddings within the shear anisotropic inhomogeneous spaces and embeddings between (classical dyadic) isotropic inhomogeneous spaces and shear anisotropic inhomogeneous spaces.

1. INTRODUCTION.

The traditional (separable) multidimensional wavelets are built from tensor products of 1-dimensional wavelets. Hence, wavelets are not very efficient in "sensing" the geometry of lower dimension discontinuities since the number of wavelets remains the same across scales. In applications it may be desirable to be able to detect more orientations having still a basis-like representation. In recent years there have been attempts to achieve this sensitivity to more orientations. Some of them include the directional wavelets or filterbanks [2], [3], the curvelets [8] and the contourlets [13], to name just a few.

In [23], Guo, Lim, Labate, Weiss and Wilson, introduced the wavelets with composite dilation. This type of representation takes full advantage of the theory of affine systems on $\mathbb{R}^d$ and therefore, unlike other representations, provides a natural transition from the continuous representation to the discrete (basis-like) setting, as in the case of wavelets. Based on the wavelets of composite dilation theory, the shearlets system provides Parseval frames for $L^2(\mathbb{R}^d)$ or subspaces of it, depending on the discrete sampling of parameters (see [23, Section 5.2] or [19]).

There are at least two other ways to define smoothness spaces with shearlets. The sophisticated theory of coorbit spaces uses auxiliary sets and spaces of functions to define Banach spaces called *shearlet coorbit spaces*, as developed by Dahlke, Häuser, Kutyniok, Steidl and Teschke in [9, 10, 11, 12]. A closer-in-spirit approach is that of Labate, Mantovani, and Negi in [27], in which a general theory of decomposition spaces is used on the shearlets system to define (quasi-)Banach spaces called *shearlet smoothness spaces*. Both of these approaches are related to Besov spaces. This can easily be seen by their definition with the shearlets coefficients in $\ell^{p,q}$ norms. One main difference of this article with the *shearlet coorbit spaces* and the *shearlet smoothness spaces* is that we define the **shear anisotropic inhomogeneous Besov and Triebel-Lizorkin *distribution* spaces** from $\ell^q(L^p)$ and $L^p(\ell^q)$ norms, respectively,

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of a sequence of convolutions of $f \in \mathcal{S}'$ with a shear anisotropic dilated functions in $\mathcal{S}$. Then, we define the associated *sequence* spaces and prove characterization. The line of argumentation follows the work of Frazier and Jawerth in [15] and [16]. In [9, 10, 11, 12] it is stated the existence of a bounded linear reconstruction operator and it is also proved the embedding of certain *shearlet coorbit spaces* into (sums of) homogeneous Besov spaces with $1 \leq p = q \leq \infty$, for a certain smoothness parameter. Regarding [27], the embeddings between the *shearlet smoothness spaces* and the classical Besov spaces are in both directions for a certain, more general, set of parameters. We prove a reproducing identity with convergence in $\mathcal{S}'$ (previously possible only for L^2). We also prove embeddings within shear anisotropic spaces and between shear anisotropic and classical spaces.

We remind the reader the definitions of classical spaces (as appear in [15] and [16]) and which will also be used in the embedding results.

Let $\varphi, \Phi \in \mathcal{S}(\mathbb{R}^d)$ satisfy

$$\text{supp } \hat{\varphi} \subseteq \{\xi \in \hat{\mathbb{R}}^d : \frac{1}{2} \leq |\xi| \leq 2\}, \qquad \text{supp } \Phi \subseteq \{\xi \in \hat{\mathbb{R}}^d : |\xi| \leq 2\}, \tag{1.1}$$

and

$$|\varphi(\xi)| \geq c > 0, \text{ if } \frac{3}{5} \leq |\xi| \leq \frac{5}{3}, \qquad |\Phi(\xi)| \geq c > 0, \text{ if } |\xi| \leq \frac{5}{3}. \tag{1.2}$$

For $\nu \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}^d$, the dyadic dilation is $\varphi_{2^\nu I}(x) := 2^{d\nu}\varphi(2^\nu x)$, where I is the identity matrix. Identifying Q with (ν, k), the dyadic dilation normalized in L^2 is $\varphi_Q = \varphi_{\nu,k} := 2^{d\nu/2}\varphi(2^\nu x)$. For $Q_0 = [0,1]^d$ let $Q_{\nu,k} = 2^{-\nu}(Q_0 + k)$. Let $\mathcal{D}_+$ denote the set of dyadic cubes $\{Q_{\nu,k} : \nu \in \mathbb{N}_0, k \in \mathbb{Z}^d\}$ and $\mathcal{D}_+^\nu$ denote the set of all dyadic cubes at scale ν, *i.e.*, $\mathcal{D}_+^\nu = \{Q_{\nu,k} : k \in \mathbb{Z}^d\}$. Let χ_Q be the characteristic function of Q and $\tilde{\chi}_Q = |Q|^{-\frac{1}{2}} \chi_Q$ the L^2-normalized characteristic function of Q.

For $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$, the inhomogeneous Besov distribution space $\mathbf{B}_p^{\alpha,q}$ is the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{\mathbf{B}_p^{\alpha,q}} = \|\Phi * f\|_{L^p} + \left(\sum_{\nu=0}^{\infty} (2^{\nu\alpha} \|\varphi_{2^\nu I} * f\|_{L^p})^q\right)^{1/q} < \infty, \tag{1.3}$$

and the inhomogeneous Besov sequence space $\mathbf{b}_p^{\alpha,q}$ is the set of all complex-valued sequences $\mathbf{s} = \{s_Q\}_{Q \in \mathcal{D}_+}$ such that

$$\|\mathbf{s}\|_{\mathbf{b}_p^{\alpha,q}} = \left(\sum_{\nu=0}^{\infty} \left(\sum_{Q \in \mathcal{D}_+^\nu} [|Q|^{-\frac{\alpha}{d}+\frac{1}{p}-\frac{1}{2}} |s_Q|]^p\right)^{q/p}\right)^{1/q} < \infty. \tag{1.4}$$

For $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$, the inhomogeneous Triebel-Lizorkin distribution space $\mathbf{F}_p^{\alpha,q}$ is the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{\mathbf{F}_p^{\alpha,q}} = \|\Phi * f\|_{L^p} + \left\| \left(\sum_{\nu=0}^{\infty} (2^{\nu\alpha} |\varphi_{2^\nu I} * f|)^q\right)^{1/q} \right\|_{L^p} < \infty, \tag{1.5}$$

and the inhomogeneous Triebel-Lizorkin sequence space $\mathbf{f}_p^{\alpha,q}$ is the set of all complex-valued sequences $\mathbf{s} = \{s_Q\}_{Q\in\mathcal{D}_+}$ such that

$$\|f\|_{\mathbf{f}_p^{\alpha,q}} = \left\| \left(\sum_{Q\in\mathcal{D}_+} (|Q|^{-\frac{\alpha}{d}} |s_Q| \tilde{\chi}_Q(\cdot))^q \right)^{1/q} \right\|_{L^p} < \infty. \tag{1.6}$$

Let ψ and Ψ be defined as φ and Φ, respectively. The main results in [15] and [16] are that the distribution spaces (1.3) and (1.5) can be characterized by the corresponding sequence spaces (1.4) and (1.6) via the analysis and synthesis operators, formally defined as

$$S_{\Phi,\varphi} f = \{\{\langle f, \Phi(\cdot - k)\rangle\}_{k\in\mathbb{Z}^d}, \{\langle f, \varphi_Q\rangle\}_{Q\in\mathcal{D}_+}\},$$

$$\text{and} \quad T_{\Psi,\psi}\mathbf{s}(\cdot) = \sum_{k\in\mathbb{Z}^d} s_k \Psi(\cdot - k) + \sum_{Q\in\mathcal{D}_+} s_Q \psi_Q(\cdot).$$

The characterization of the Triebel-Lizorkin spaces in [16] uses the next definitions (which will be used for the embedding results): identify Q and P with (ν, k) and (ν, k'), respectively. For all $r > 0$, $N \in \mathbb{N}$ and $\nu \geq 0$, define

$$(s^*_{r,N})_Q := \left(\sum_{Q\in\mathcal{D}_+} \frac{|s_P|^r}{(1 + 2^\nu |x_Q - x_P|)^N} \right)^{1/r},$$

where $x_Q = 2^{-\nu}k$ is the "lower left corner" of $Q_{\nu,k}$. Define also $\mathbf{s}^*_{r,N} := \{(s^*_{r,N})_Q\}_{Q\in\mathcal{D}_+}$. With an additional assumption on the functions ψ, Ψ, φ and Φ, a reproducing identity with convergence in $\mathcal{S}'$ is also proved in [15, Lemma 2.1]. This means that $f = T_{\Psi,\psi} \circ S_{\Phi,\phi} f$ with convergence in $\mathcal{S}'$. We remind the reader that our notation is slightly different from [15] and [16] with respect to the dilations and, therefore, our notation for the analysis and synthesis operators is different from [15] and [16] (see (2.14) in [15]).

The reader can now compare (1.3) and (1.4) with (4.1) and (4.2) for the Besov-like spaces and (1.5) and (1.6) with (5.1) and (5.2) for the Triebel-Lizorkin-like spaces. See Section 2 for the definitions of shear anisotropic dilations.

Some results for shear anisotropic inhomogeneous Triebel-Lizorkin spaces for $d = 2$ appeared first in [33].

The outline of the paper is as follows. In Section 2 we introduce the "shearlets on the cone" system and set notation. In Section 3 we give two basic lemmata regarding almost orthogonality and present classical results slightly modified. In Sections 4 and 5 we define the shear anisotropic inhomogeneous (Besov and Triebel-Lizorkin, respectively) distributions and sequence spaces and prove characterization. A reproducing identity in $\mathcal{S}'$ is proved in Section 6. In Section 7 we prove Sobolev-type embeddings within the shear anisotropic inhomogeneous spaces and embeddings between (classical dyadic) isotropic inhomogeneous spaces and shear anisotropic inhomogeneous spaces for certain smoothness parameters. In this section we also prove that there exist sequences of non-vanishing functions in one of the classical or shear anisotropic spaces that vanish in the norm of the other space. Proofs for Section 3 and for some results in Section 5 are given in Section 8.

2. Shearlets on the cone and notation

Define the cone aligned with the ξ_1 axis as

$$\mathcal{D}^{(1)} = \{(\xi_1, \dots, \xi_d) \in \hat{\mathbb{R}}^d : |\xi_1| \geq \frac{1}{8}, \left|\frac{\xi_\mathfrak{d}}{\xi_1}\right| \leq 1, \mathfrak{d} = 2, \dots, d\}. \tag{2.1}$$

Let $\hat{\psi}_1, \hat{\psi}_2 \in C^\infty(\mathbb{R})$ with supp $\hat{\psi}_1 \subset [-\frac{1}{2}, -\frac{1}{16}] \cup [\frac{1}{16}, \frac{1}{2}]$ and supp $\hat{\psi}_2 \subset [-1, 1]$ such that

$$\sum_{j\geq 0} \left|\hat{\psi}_1(2^{-2j}\omega)\right|^2 = 1, \quad \text{for } |\omega| \geq \frac{1}{8} \tag{2.2}$$

and

$$\left|\hat{\psi}_2(\omega - 1)\right|^2 + \left|\hat{\psi}_2(\omega)\right|^2 + \left|\hat{\psi}_2(\omega + 1)\right|^2 = 1, \quad \text{for } |\omega| \leq 1. \tag{2.3}$$

It follows from (2.3) that, for $j \geq 0$,

$$\sum_{\ell=-2^j}^{2^j} \left|\hat{\psi}_2(2^j\omega - \ell)\right|^2 = 1, \quad \text{for } |\omega| \leq 1. \tag{2.4}$$

For a scale $j \geq 0$ the anisotropic dilation matrices are defined as

$$A^j_{(1)} = \begin{pmatrix} 4^j & 0 & \dots & 0 \\ 0 & 2^j & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 2^j \end{pmatrix}, \quad \dots \quad, A^j_{(d)} = \begin{pmatrix} 2^j & 0 & \dots & 0 \\ 0 & 2^j & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 4^j \end{pmatrix},$$

and for $\ell = (\ell_1, \dots, \ell_{d-1})$ with $-2^j \leq \ell_i \leq 2^j$, $i = 1, \dots, d-1$, the $d \times d$ shear matrices are defined as

$$B^{[\ell]}_{(1)} = \begin{pmatrix} 1 & \ell_1 & \dots & \ell_{d-1} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \quad \dots, B^{[\ell]}_{(d)} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \ell_1 & \ell_2 & \dots & 1 \end{pmatrix}.$$

To shorten notation we will write $|[\ell]| \preceq 2^j$ instead of $|\ell_i| \leq 2^j$, $i = 1, \dots, d-1$, and $|[\ell]| = 2^j$ when $|\ell_i| = 2^j$ for at least one $i = 1, 2, \dots, d-1$. Define $\hat{\psi}^{(1)}(\xi) := \hat{\psi}_1(\xi_1) \prod_{\mathfrak{d}=2}^d \hat{\psi}_2(\frac{\xi_\mathfrak{d}}{\xi_1})$. Since $\xi A^{-j}_{(1)} B^{[-\ell]}_{(1)} = (4^{-j}\xi_1, -4^{-j}\xi_1\ell_1 + 2^{-j}\xi_2, \dots, -4^{-j}\xi_1\ell_{d-1} + 2^{-j}\xi_d)$, from (2.2) and (2.4) it follows that

$$\begin{aligned} &\sum_{j\geq 0} \sum_{|[\ell]|\preceq 2^j} \left|\hat{\psi}^{(1)}(\xi A^{-j}_{(1)} B^{[-\ell]}_{(1)})\right|^2 \\ &\qquad = \sum_{j\geq 0} \sum_{|\ell_1|,\dots,|\ell_{d-1}|\leq 2^j} \left|\hat{\psi}_1(2^{-2j}\xi_1)\right|^2 \prod_{\mathfrak{d}=2}^d \left|\hat{\psi}_2(2^j \frac{\xi_\mathfrak{d}}{\xi_1} - \ell_{\mathfrak{d}-1})\right|^2 \\ &\qquad = \sum_{j\geq 0} \left|\hat{\psi}_1(2^{-2j}\xi_1)\right|^2 \sum_{|\ell_1|,\dots,|\ell_{d-2}|\leq 2^j} \prod_{\mathfrak{d}=2}^{d-1} \left|\hat{\psi}_2(2^j \frac{\xi_\mathfrak{d}}{\xi_1} - \ell_{\mathfrak{d}-1})\right|^2 \\ &\qquad \vdots \\ &\qquad = 1, \end{aligned} \tag{2.5}$$

for $\xi = (\xi_1, \ldots, \xi_d) \in \mathcal{D}^{(1)}$ and which we will call the **Parseval frame condition** (for the cone $\mathcal{D}^{(1)}$). Since supp $\hat{\psi}^{(1)} \subset [-\frac{1}{2}, \frac{1}{2}]^d$, (2.5) implies that the shearlet system

$$\{\psi^{(1)}_{j,\ell,k}(x) = \left|\det A_{(1)}\right|^{j/2} \psi^{(1)}(B^{[\ell]}_{(1)} A^{j}_{(1)} x - k) : j \geq 0, |[\ell]| \preceq 2^j, k \in \mathbb{Z}^d\}, \tag{2.6}$$

is a Parseval frame for $L^2((\mathcal{D}^{(1)})^\vee) = \{f \in L^2(\mathbb{R}^d) : \text{supp } \hat{f} \subset \mathcal{D}^{(1)}\}$ (see [23], Subsection 5.2.1). This means that

$$\sum_{j\geq 0} \sum_{|[\ell]|\preceq 2^j} \sum_{k\in\mathbb{Z}^d} \left|\langle f, \psi^{(1)}_{j,\ell,k}\rangle\right|^2 = \|f\|^2_{L^2(\mathbb{R}^d)},$$

for all $f \in L^2(\mathbb{R}^d)$ such that supp $\hat{f} \subset \mathcal{D}^{(1)}$. There are several examples of functions ψ_1, ψ_2 satisfying the properties described above (see [20]). Since $\hat{\psi}^{(1)} \in C^\infty_c(\hat{\mathbb{R}}^d)$, there exists C_N such that $\left|\psi^{(1)}(x)\right| \leq C_N(1+|x|)^{-N}$ for all $N \in \mathbb{N}$. The geometric properties of the shearlets system in $\mathcal{D}^{(1)}$ are more evident by observing that

$$\text{supp } (\psi^{(1)}_{j,\ell,k})^\wedge \subset \{\xi \in \hat{\mathbb{R}}^d : |\xi_1| \in [2^{2j-4}, 2^{2j-1}], \left|2^j \frac{\xi_{\mathfrak{d}}}{\xi_1} - \ell_{\mathfrak{d}-1}\right| \leq 1, \mathfrak{d} = 2, \ldots, d\}.$$

One can also construct a shearlets system for any cone

$$\mathcal{D}^{(i)} = \{\xi = (\xi_1, \ldots, \xi_i, \ldots, \xi_d) \in \hat{\mathbb{R}}^d : |\xi_i| \geq \frac{1}{8}, \left|\frac{\xi_{\mathfrak{d}}}{\xi_i}\right| \leq 1, \mathfrak{d} \neq i\},$$

by defining $\hat{\psi}^{(i)}(\xi) = \hat{\psi}_1(\xi_i) \prod_{\mathfrak{d}\neq i} \hat{\psi}_2(\frac{\xi_{\mathfrak{d}}}{\xi_i})$ and choosing correspondingly the anisotropic and shear matrices $A^j_{(i)}$ and $B^{[\ell]}_{(i)}$.

Let $\hat{\Psi} \in C^\infty_c(\mathbb{R}^d)$, with supp $\hat{\Psi} \subset [-\frac{1}{4}, \frac{1}{4}]^d$ and $\left|\hat{\Psi}\right| = 1$ for $\xi \in [-\frac{1}{8}, \frac{1}{8}]^d = \mathcal{R}$, be such that

$$\left|\hat{\Psi}(\xi)\right|^2 \chi_{\mathcal{R}}(\xi) + \sum_{\mathfrak{d}=1}^{d} \sum_{j\geq 0} \sum_{|[\ell]|\preceq 2^j} \left|\hat{\psi}^{(\mathfrak{d})}(\xi A^{-j}_{(\mathfrak{d})} B^{[-\ell]}_{(\mathfrak{d})})\right|^2 \chi_{\mathcal{D}^{(\mathfrak{d})}}(\xi) = 1, \tag{2.7}$$

for all $\xi \in \hat{\mathbb{R}}^d$. This implies that one can construct a Parseval frame for $L^2(\mathbb{R}^d)$, see [28, Theorem 9].

Since $\mathcal{D}^{(i)}$ are orthogonal rotations of $\mathcal{D}^{(1)}$ we will develop our results only for direction 1. We will incorporate the directions only in the definitions of the spaces. Then, dropping the subindices in the matrices and the superindices in the shearlet functions we have

$$B^{[\ell]} A^j x = \begin{pmatrix} 2^{2j} x_1 + 2^j \ell_1 x_2 + \ldots + 2^j \ell_{d-1} x_d \\ 2^j x_2 \\ \vdots \\ 2^j x_d \end{pmatrix}, \tag{2.8}$$

for every $j \geq 0$ and $|[\ell]| \preceq 2^j$ and $\psi_{j,\ell,k}(x) = |\det A|^{j/2} \psi(B^{[\ell]} A^j x - k)$.

2.1. **Notation.** We denote for a matrix $M \in GL_d(\mathbb{R})$ the anisotropic dilation $\psi_M(x) = |\det M|^{-1}\psi(M^{-1}x)$. We also denote $\tilde{\psi}(x) = \overline{\psi(-x)}$. For $Q_0 = [0,1)^d$, write

$$Q_{j,\ell,k} = A^{-j}B^{-[\ell]}(Q_0 + k), \tag{2.9}$$

with $j \geq 0$, $|[\ell]| \preceq 2^j$ and $k \in \mathbb{Z}^d$. Therefore, $\int \chi_{Q_{j,\ell,k}} = |Q_{j,\ell,k}| = 2^{-(d+1)j} = |\det A|^{-j}$. We also write $\tilde{\chi}_Q(x) = |Q|^{-1/2}\chi_Q(x)$. Let $\mathcal{Q}_{AB} := \{Q_{j,\ell,k} : j \geq 0, |[\ell]| \preceq 2^j, k \in \mathbb{Z}^d\}$ and $\mathcal{Q}^{j,\ell} := \{Q_{j,\ell,k} : k \in \mathbb{Z}^d\}$. Then, $\mathcal{Q}^{j,\ell}$ is a partition of $\mathbb{R}^d$. To shorten notation and clear exposition, we will identify the multi indices (j,ℓ,k) and (i,m,n) with P and Q, respectively. This way we write $\psi_P = \psi_{j,\ell,k}$ or $\psi_Q = \psi_{i,m,n}$. Also, let x_P and x_Q be the "lower left corners" $A^{-j}B^{-[\ell]}k$ and $A^{-i}B^{-[m]}n$ of the "cubes" $P = Q_{j,\ell,k}$ and $Q = Q_{i,m,n}$, respectively. Let $B_r(x)$ be the Euclidean ball centered in x with radius r.

The elements of the shearlets system

$$\{\psi_{j,\ell,k}(x) = |\det A|^{j/2}\psi(B^{[\ell]}A^j x - k) : j \geq 0, |[\ell]| \preceq 2^j, k \in \mathbb{Z}^d\},$$

have Fourier transform

$$(\psi_{j,\ell,k})^\wedge(\xi) = |\det A|^{-j/2}\hat{\psi}(\xi A^{-j}B^{-[\ell]})\mathbf{e}^{-2\pi i\xi A^{-j}B^{[-\ell]}k}.$$

Using the anisotropic dilation it is also easy to verify that

$$\psi_{A^{-j}B^{-[\ell]}}(x - A^{-j}B^{-[\ell]}k) = |\det A|^{j/2}\psi_{j,\ell,k}(x) = |P|^{-1/2}\psi_P(x),$$

and thus

$$\left(\psi_{A^{-j}B^{-[\ell]}}(\cdot - A^{-j}B^{-[\ell]}k)\right)^\wedge(\xi) = \hat{\psi}(\xi A^{-j}B^{-[\ell]})\mathbf{e}^{-2\pi i\xi A^{-j}B^{-[\ell]}k}.$$

We also have

$$\begin{aligned} \langle f, \psi_P\rangle &= \langle f, \psi_{j,\ell,k}\rangle \\ &= \int_{\mathbb{R}^d} f(x)\overline{|\det A|^{-j/2}\psi_{A^{-j}B^{[-\ell]}}(x - A^{-j}B^{[-\ell]}k)}dx \\ &= |P|^{1/2}(f * \tilde{\psi}_{A^{-j}B^{[-\ell]}})(x_P). \end{aligned} \tag{2.10}$$

We formally define the **analysis** and **synthesis operators** as

$$S_{\Psi,\psi}f = \{\{\langle f, \Psi(\cdot - k)\rangle\}_{k\in\mathbb{Z}^d}, \{\langle f, \psi_Q\rangle\}_{Q\in\mathcal{Q}_{AB}}\} \tag{2.11}$$

and

$$T_{\Psi,\psi}\mathbf{s} = \sum_{k\in\mathbb{Z}^d} s_k\Psi(\cdot - k) + \sum_{Q\in\mathcal{Q}_{AB}} s_Q\psi_Q, \tag{2.12}$$

respectively. We remind the reader that the function related to the low frequencies Ψ (see (2.7)) and associated sequence $\{s_k\}_{k\in\mathbb{Z}^d}$ are not studied in this work since they are already treated in the literature (see *e.g.* [15, Section 7] or [16, Section 12]).

3. Basic results

3.1. **Almost orthogonality.** The next two "almost orthogonality" results are in the form of convolutions in the time domain. The first is between two functions with shear anisotropic dilations and is used in both characterizations. The second is between a function with shear anisotropic dilation and other function with dyadic dilation and is used in the embeddings. The last "almost orthogonality" result is in the Fourier

domain and is used in both characterizations. The three of these results are proved in Section 8.1.

Lemma 3.1. *Let $g, h \in \mathcal{S}$. For $i = j-1,\ j,\ j+1 \geq 0$, let Q be identified with (i, m, n). Then, for every $N > d$, there exists a $C_N > 0$ such that*

$$|g_{A^{-j}B^{-\ell}} * h_Q(x)| \leq \frac{C_N\,|Q|^{-\frac{1}{2}}}{(1 + 2^i\,|x - x_Q|)^N},$$

for all $x \in \mathbb{R}^d$.

Lemma 3.2. *Let $\psi, \varphi \in \mathcal{S}$. For $j \geq 0$, $|[\ell]| \preceq 2^j$ and $k \in \mathbb{Z}^d$,*

$$\int_{\mathbb{R}^d} \left|\psi(B^{[\ell]}A^j(x-y))\right| \left|\varphi(2^{2j}y)\right| dy \leq \frac{C_N\,|P_j|}{(1+2^j\,|x|)^N},$$

for all $N > d$.

Lemma 3.3. *Let $(\psi_{j,\ell,k})^\wedge$ be as in the introduction of this section. Then, the support of $(\psi_{j,\ell,k})^\wedge$ overlaps with the support of at most $2^{(d-1)} + 3^{(d-1)} + 6^{(d-1)}$ other shearlets $(\psi_{i,m,n})^\wedge$ for $(j,\ell) \neq (i,m)$ and all $k, n \in \mathbb{Z}^d$.*

Remark 3.4. *Since the translation parameters k and n do not affect the support in the frequency domain, then for*

$$f = T_\psi \mathbf{s} = \sum_{Q \in \mathcal{Q}_{AB}} s_Q \psi_Q = \sum_{i \geq 0} \sum_{|[m]| \preceq 2^i} \sum_{n \in \mathbb{Z}^d} s_{i,m,n} \psi_{i,m,n},$$

we formally have that

$$(\tilde{\psi}_{A^{-j}B^{[-\ell]}} * f)(x) = \sum_{i=j-1}^{j+1} \sum_{m(\ell,i)} \sum_{Q \in \mathcal{Q}^{i,m}} s_Q (\tilde{\psi}_{A^{-j}B^{[-\ell]}} * \psi_Q)(x),$$

where $m(\ell, i)$ are the shear indices of those shearlets in the Fourier domain "surrounding" the support of $(\tilde{\psi}_{A^{-j}B^{[-\ell]}})^\wedge$ and the sum $\sum_{i=j-1}^{j+1} \sum_{m(\ell,i)}$ has at most $3^{(d-1)} + 3^{(d-1)} + 6^{(d-1)} + 1$ terms for all (j,ℓ) by Lemma 3.3.

Remark 3.5. *From Lemma 3.3 the number of shearlets in other cones overlapping (even with shearlets within different cones) on the Fourier domain is bounded for all scales, since their respective systems are orthonormal rotations of the system for $\mathcal{D}^1$. Therefore, removing the characteristic functions $\chi_{\mathcal{D}^{(0)}}$ and $\chi_{\mathcal{R}}$ in (2.7) affects only the Parseval condition on the frame.*

Remark 3.6. *The notion of almost orthogonality has been used in the context of shearlets in* [21] *to bound the magnitude of the inner product of more general* shearlet molecules *using a dyadic parabolic pseudo-distance. One consequence is that more general frames can be defined by non band-limited shearlet-like functions as in* [27]. *In this article we only use the Euclidean distance and prove a reproducing identity with the "smooth Parseval frames of shearlets".*

3.2. **Shear anisotropic dilations and classical results.** The next two Lemmata will be used to prove the characterization of the shear anisotropic inhomogeneous Besov spaces. Their proof are in Section 8.1 and are just variations of those found in [17], we include them for completeness.

Lemma 3.7. ***[Sampling lemma]*** *Let $g \in \mathcal{S}'$ and $h \in \mathcal{S}$ be such that*

$$\operatorname{supp}\, \hat{g},\ \operatorname{supp}\, \hat{h} \subset [-1/2, 1/2]^d B^{[\ell]} A^j, \qquad j \geq 0, |[\ell]| \preceq 2^j.$$

Then,

$$g * h = \sum_{k \in \mathbb{Z}^d} |\det\, A|^{-j}\, g(A^{-j} B^{[-\ell]} k) h(x - A^{-j} B^{[-\ell]} k),$$

with convergence in $\mathcal{S}'$.

Lemma 3.8. ***[Plancherel-Pólya]*** *Let $0 < p \leq \infty$ and $j \geq 0$. Suppose $g \in \mathcal{S}'$ and* $\operatorname{supp}\, \hat{g} \subseteq [-\frac{1}{2}, \frac{1}{2}]^d B^{[\ell]} A^j$. *Then,*

$$\left(\sum_{Q \in \mathcal{Q}^{j,\ell}} \sup_{z \in Q} |g(z)|^p \right)^{1/p} \leq C_p\, |Q_j|^{-\frac{d}{p(d+1)}}\, \|g\|_{L^p}.$$

The next definition and result are well known and will be used to characterize the shear anisotropic inhomogeneous Triebel-Lizorkin distribution spaces.

Definition 3.9. *The* ***Hardy-Littlewood maximal function****, $\mathcal{M}f(x)$, is given by*

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\, dy,$$

for a locally integrable function f on $\mathbb{R}^d$ and where $B_r(x)$ is the ball with center in x and radius r.

It is well known that $\mathcal{M}$ is bounded on L^p, $1 < p \leq \infty$. It is also true that the next vector-valued inequality holds (see [14]).

Theorem 3.10. ***[Fefferman-Stein]*** *For $1 < p < \infty$ and $1 < q \leq \infty$, there exist a constant $C_{p,q}$ such that*

$$\left\| \left\{ \sum_{i=1}^{\infty} (\mathcal{M}f_i)^q \right\}^{1/q} \right\|_{L^p} \leq C_{p,q} \left\| \left\{ \sum_{i=1}^{\infty} f_i^q \right\}^{1/q} \right\|_{L^p},$$

for any sequence $\{f_i : i = 1, 2, \ldots\}$ of locally integrable functions.

4. Shear anisotropic inhomogeneous Besov spaces

After defining the spaces we will leave aside the analysis of the low frequencies and the analysis for all directions. We follow [15].

Definition 4.1. *Let $\hat{\Psi}$ and $\hat{\psi}$ be such that the Parseval frame condition (2.7) is satisfied. For $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$, the* ***shear anisotropic inhomogeneous Besov*** **distribution** ***space*** *is defined as the collection of all $f \in \mathcal{S}'$ such that*

$$\|f\|_{\mathbf{B}_p^{\alpha,q}(AB)} := \|f * \Psi\|_{L^p} + \left(\sum_{\mathfrak{d}=1}^{d} \sum_{j\geq 0} \sum_{|[\ell]|\preceq 2^j} [|Q_j|^{-\alpha} \left\| f * \psi^{(\mathfrak{d})}_{A^{-j}_{(\mathfrak{d})} B^{-\ell}_{(\mathfrak{d})}} \right\|_{L^p}]^q \right)^{1/q} < \infty. \quad (4.1)$$

Definition 4.2. *For $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$, the* ***shear anisotropic inhomogeneous Besov*** **sequence** ***space*** *is defined as the collection of all complex-valued sequences $\mathbf{s} = \{s_Q\}_{Q \in \mathcal{Q}_{AB}}$ such that*

$$\|\mathbf{s}\|_{\mathbf{b}_p^{\alpha,q}(AB)} := \left(\sum_{k\in\mathbb{Z}^d} |s_k|^p\right)^{1/p} + \left(\sum_{\mathfrak{d}=1}^{d} \sum_{j\geq 0} \sum_{|[\ell]|\preceq 2^j} \left(\sum_{Q\in\mathcal{Q}^{j,\ell}} [|Q|^{-\alpha+\frac{d}{p(d+1)}-\frac{1}{2}} |s_Q|]^p\right)^{q/p}\right)^{1/q} < \infty. \quad (4.2)$$

4.1. **The characterization.** We prove the boundedness of the analysis and synthesis operators (2.11) and (2.12) on the spaces in Definitions 4.1 and 4.2.

Theorem 4.3. *Let $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$. Then, the operators $S_\psi : \mathbf{B}_p^{\alpha,q}(AB) \to \mathbf{b}_p^{\alpha,q}(AB)$ and $T_\psi : \mathbf{b}_p^{\alpha,q}(AB) \to \mathbf{B}_p^{\alpha,q}(AB)$ are well defined and bounded.*

Proof. We prove only the case $p, q < \infty$. To prove the boundedness of S_ψ assume $f \in \mathbf{B}_p^{\alpha,q}(AB)$. From Lemma 3.7 we have

$$\begin{aligned}
& f * \tilde{\psi}_{A^{-j}B^{-\ell}} * \psi_{A^{-j}B^{-\ell}}(x) \\
&= \sum_{k\in\mathbb{Z}^d} |\det A|^{-j} f * \tilde{\psi}_{A^{-j}B^{-\ell}}(A^{-j}B^{-\ell}k) \cdot \psi_{A^{-j}B^{-\ell}}(x - A^{-j}B^{-\ell}k) \\
&= \sum_{k\in\mathbb{Z}^d} |\det A|^{-\frac{j}{2}} f * \tilde{\psi}_{A^{-j}B^{-\ell}}(A^{-j}B^{-\ell}k) \cdot \psi_{j,\ell,k}(x),
\end{aligned}$$

with convergence in $\mathcal{S}'$. Identify Q with (j, ℓ, k). Then, $s_Q = \langle f, \psi_{j,\ell,k}\rangle = |Q|^{\frac{1}{2}} f * \tilde{\psi}_{A^{-j}B^{-\ell}}(A^{-j}B^{-\ell}k)$ (see (2.10)) and Lemma 3.8 yields

$$\begin{aligned}
& \left(\sum_{Q\in\mathcal{Q}^{j,\ell}} [|Q|^{-\alpha+\frac{d}{p(d+1)}-\frac{1}{2}} |Q|^{\frac{1}{2}} \left| f * \tilde{\psi}_{A^{-j}B^{-\ell}}(A^{-j}B^{-\ell}k)\right|]^p\right)^{1/p} \\
&\qquad \leq C_p |Q|^{-\frac{d}{p(d+1)}} \left\| |Q|^{-\alpha+\frac{d}{p(d+1)}} f * \tilde{\psi}_{A^{-j}B^{-\ell}} \right\|_{L^p}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathbf{s}\|_{\mathbf{b}_p^{\alpha,q}(AB)} &\leq C_p \left(\sum_{\mathfrak{d}} \sum_{j\geq 0} \sum_{|[\ell]|\preceq 2^j} [|Q|^{-\alpha} \left\| f * \tilde{\psi}_{A^{-j}B^{-\ell}} \right\|_{L^p}]^q \right)^{1/q} \\
&= C_p \|f\|_{\mathbf{B}_p^{\alpha,q}(AB)}.
\end{aligned}$$

To prove the boundedness of T_ψ assume $\mathbf{s} \in \mathbf{b}_p^{\alpha,q}(AB)$. Identify Q with (i,m,n) and let $f = \sum_{Q\in\mathcal{Q}_{AB}} s_Q\psi_Q$. By Lemma 3.3, Remark 3.4 and Lemma 3.1,

$$\|\psi_{A^{-j}B^{[-\ell]}} * f\|_{L^p} \leq C_p \sum_{i=j-1}^{j+1} \sum_{m(\ell,i)} \left(\int_{\mathbb{R}^d} (\sum_{Q\in\mathcal{Q}^{i,m}} \frac{|s_Q|\,|Q|^{-\frac{1}{2}}}{(1+2^j\,|x-x_Q|)^N})^p dx \right)^{1/p},$$

for all $N > d$. By the p-triangular inequality $|a+b|^p \leq |a|^p + |b|^p$ if $0 < p \leq 1$ or by Hölder's inequality if $1 < p < \infty$, for a sufficiently large N we have

$$\|\psi_{A^{-j}B^{[-\ell]}} * f\|_{L^p} \leq C_p \sum_{i=j-1}^{j+1} \sum_{m(\ell,i)} \left(\int_{\mathbb{R}^d} \sum_{Q\in\mathcal{Q}^{i,m}} \frac{|s_Q|^p\,|Q|^{-\frac{p}{2}}}{(1+2^j\,|x-x_Q|)^{d+1}})^p dx \right)^{1/p}.$$

Therefore, (notice that, since $|i-j| \leq 1$, the change from $Q \in \mathcal{Q}^{i,m}$ to $Q \in \mathcal{Q}^{j,\ell}$ is harmless)

$$\begin{aligned} \|f\|_{\mathbf{B}_p^{\alpha,q}(AB)} &\leq C_{p,q} \left(\sum_{\mathfrak{d}=1}^{d} \sum_{j\geq 0} \sum_{|[\ell]|\preceq 2^j} |Q_j|^{-\alpha q} \left[\sum_{Q\in\mathcal{Q}^{j,\ell}} |s_Q|^p\,|Q|^{-\frac{p}{2}+\frac{d}{(d+1)}} \right]^{\frac{q}{p}} \right)^{1/q} \\ &= C_{p,q}\,\|\mathbf{s}\|_{\mathbf{b}_p^{\alpha,q}(AB)}, \end{aligned}$$

which is what we wanted to prove. ■

Remark 4.4. *With the same arguments as in Remark 2.6 in* [16]*, the definition of* $\mathbf{B}_p^{\alpha,q}(AB)$ *is independent of the choice of* $\psi \in \mathcal{S}$ *as long as it satisfies the requirements in Section 2.*

5. Shear anisotropic inhomogeneous Triebel-Lizorkin spaces

After defining the spaces we will leave aside the analysis of the low frequencies and the analysis for all directions. We follow [16].

Definition 5.1. *Let* $\Psi, \psi \in \mathcal{S}$ *be as in Section 2. Let* $\alpha \in \mathbb{R}$, $0 < p < \infty$ *and* $0 < q \leq \infty$. *The* ***shear anisotropic inhomogeneous Triebel-Lizorkin* distribution *space*** $\mathbf{F}_p^{\alpha,q}(AB)$ *is defined as the collection of all* $f \in \mathcal{S}'$ *such that*

$$\begin{aligned} \|f\|_{\mathbf{F}_p^{\alpha,q}(AB)} &= \|f * \Psi\|_{L^p} \\ &+ \left\| \left(\sum_{\mathfrak{d}=1}^{d} \sum_{j\geq 0} \sum_{\ell=-2^j}^{2^j} [|Q_j|^{-\alpha} \left| \tilde{\psi}^{\mathfrak{d}}_{A_{(\mathfrak{d})}^{-j} B_{(\mathfrak{d})}^{-\ell}} * f \right|]^q \right)^{1/q} \right\|_{L^p} < \infty. \end{aligned} \tag{5.1}$$

Definition 5.2. *Let* $\alpha \in \mathbb{R}$, $0 < p < \infty$ *and* $0 < q \leq \infty$. *The* ***shear anisotropic inhomogeneous Triebel-Lizorkin* sequence *space*** $\mathbf{f}_p^{\alpha,q}(AB)$ *is defined as the collection of all complex-valued sequences* $\mathbf{s} = \{s_Q\}_{Q\in\mathcal{Q}_{AB}}$ *such that*

$$\|\mathbf{s}\|_{\mathbf{f}_p^{\alpha,q}(AB)} = \left\| \left(\sum_{Q\in\mathcal{Q}_{AB}} (|Q|^{-\alpha}\,|s_Q|\,\tilde{\chi}_Q)^q \right)^{1/q} \right\|_{L^p} < \infty. \tag{5.2}$$

5.1. **Two basic results.** The proof of the characterization follows the corresponding result in [16]. This is based on a kind of Peetre's inequality to bound $S_\psi : \mathbf{F}_p^{\alpha,q}(AB) \to \mathbf{f}_p^{\alpha,q}(AB)$, and a characterization of $\mathbf{f}_p^{\alpha,q}(AB)$ to bound $T_\psi : \mathbf{f}_p^{\alpha,q}(AB) \to \mathbf{F}_p^{\alpha,q}(AB)$.

For all $\lambda > 0$, let

$$(\psi^{**}_{j,\ell,\lambda} f)(x) := \sup_{y\in\mathbb{R}^d} \frac{|(\psi_{A^{-j}B^{-\ell}} * f)(x-y)|}{(1+|B^\ell A^j y|)^{d\lambda}}, \tag{5.3}$$

be the ***shear anisotropic*** **Peetre's maximal function**. The next result can be regarded as a ***shear anisotropic*** **Peetre's inequality** and is proved in Section 8.2.

Lemma 5.3. *Let ψ be band limited, $f \in \mathcal{S}'$, $j \geq 0$ and $\|[\ell]\| \preceq 2^j$. Then, for any real $\lambda > 0$, there exists a constant C_λ such that*

$$(\psi^{**}_{j,\ell,\lambda} f)(x) \leq C_\lambda \left\{ \mathcal{M}(|\psi_{A^{-j}B^{-\ell}} * f|^{1/\lambda})(x) \right\}^\lambda, \quad x \in \mathbb{R}^d.$$

Identify Q and P with (i,m,n) and (j,ℓ,k), respectively. For all $r>0$, $N \in \mathbb{N}$ and $i \geq j \geq 0$, define

$$(s^*_{r,N})_Q := \left(\sum_{P\in\mathcal{Q}^{j,\ell}} \frac{|s_P|^r}{(1+2^j\,|x_Q - x_P|)^N} \right)^{1/r},$$

and $\mathbf{s}^*_{r,N} = \{(s^*_{r,N})_Q\}_{Q\in\mathcal{Q}_{AB}}$. We then have the characterization of the sequence spaces $\mathbf{f}_p^{\alpha,q}(AB)$ in terms of $\mathbf{s}^*_{r,N}$. Next result is also proved in Section 8.2.

Lemma 5.4. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Then, for all $r > 0$ and $N > (d+1)\max(1, r/q, r/p)$ there exists $C_{r,d} > 0$ such that*

$$\|\mathbf{s}\|_{\mathbf{f}_p^{\alpha,q}(AB)} \leq \left\|\mathbf{s}^*_{r,N}\right\|_{\mathbf{f}_p^{\alpha,q}(AB)} \leq C_{r,d}\,\|\mathbf{s}\|_{\mathbf{f}_p^{\alpha,q}(AB)}.$$

5.2. **The characterization.** We prove the boundedness of the analysis and synthesis operators (2.11) and (2.12) on the spaces in Definitions 5.1 and 5.2.

Theorem 5.5. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Then, the operators $S_\psi : \mathbf{F}_p^{\alpha,q}(AB) \to \mathbf{f}_p^{\alpha,q}(AB)$ and $T_\psi : \mathbf{f}_p^{\alpha,q}(AB) \to \mathbf{F}_p^{\alpha,q}(AB)$ are well defined and bounded.*

Proof. We prove only the case $q < \infty$. To prove the boundedness of S_ψ suppose $f \in \mathbf{F}_p^{\alpha,q}(AB)$. Let P be identified with (j,ℓ,k). Then, $\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f(x_P)\right| \chi_P = |\langle f, \psi_P\rangle|\,\tilde{\chi}_P$, as in (2.10). Let E be the set of parallelograms in $\mathcal{Q}^{j,\ell}$ surrounding the origin. Since $\mathcal{Q}^{j,\ell}$ is a partition of $\mathbb{R}^d$ we have for $x \in P$ with $P \in \mathcal{Q}^{j,\ell}$,

$$\begin{aligned}
&\sum_{P\in\mathcal{Q}^{j,\ell}} [|P|^{-\alpha}\,|(S_\psi f)_P|\,\tilde{\chi}_P(x)]^q \\
&\quad= |\det A|^{j\alpha q} \sum_{P\in\mathcal{Q}^{j,\ell}} \left[\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f(x_P)\right| \chi_P(x)\right]^q \\
&\quad\leq |\det A|^{j\alpha q} \sum_{P\in\mathcal{Q}^{j,\ell}} \sup_{y\in P} \left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f(y)\right|^q \chi_P(x) \\
&\quad\leq |\det A|^{j\alpha q} \sup_{z\in E} \left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f(x-z)\right|^q
\end{aligned}$$

$$
\begin{aligned}
&= |\det A|^{j\alpha q} \sup_{z\in E}\left[\frac{\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f(x-z)\right|}{(1+|B^{\ell}A^{j}z|)^{d/\lambda}}\right]^{q}(1+\left|B^{\ell}A^{j}z\right|)^{qd/\lambda} \\
&\le |\det A|^{j\alpha q}\left[\sup_{z\in\mathbb{R}^d}\frac{\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f(x-z)\right|}{(1+|B^{\ell}A^{j}z|)^{d/\lambda}}\right]^{q}(1+\mathrm{Diam}(Q_{0,0,1}))^{qd/\lambda} \\
&= C_{d,q,\lambda}\,|\det A|^{j\alpha q}\,(\tilde{\psi}^{**}_{j,\ell,1/\lambda}f)^{q}(x) \\
&\le C_{d,q,\lambda}\,|\det A|^{j\alpha q}\left\{\mathcal{M}\left(\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f\right|^{\lambda}\right)(x)\right\}^{q/\lambda},
\end{aligned}
$$

because of Lemma 5.3 (with $1/\lambda$ instead of λ in the last inequality). Now, take $0<\lambda<\min(p,q)$. Then, the previous estimate and Theorem 3.10 yield

$$
\begin{aligned}
\|S_\psi f\|_{\mathbf{f}^{\alpha,q}_p(AB)} &= \left\|\left(\sum_{j\ge 0}\sum_{[\ell]\preceq 2^j}\sum_{P\in\mathcal{Q}^{j,\ell}}[|P|^{-\alpha}\,|(S_\psi f)_P|\,\tilde{\chi}_P]^{q}\right)^{1/q}\right\|_{L^p} \\
&\le C_{d,q,\lambda}\left\|\left(\sum_{j\ge 0}\sum_{[\ell]\preceq 2^j}\left\{\mathcal{M}\left(|\det A|^{j\alpha\lambda}\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f\right|^{\lambda}\right)\right\}^{q/\lambda}\right)^{1/q}\right\|_{L^p} \\
&= C_{d,q,\lambda}\left\|\left(\sum_{j\ge 0}\sum_{[\ell]\preceq 2^j}\left\{\mathcal{M}\left(|\det A|^{j\alpha\lambda}\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f\right|^{\lambda}\right)\right\}^{q/\lambda}\right)^{\lambda/q}\right\|_{L^{p/\lambda}}^{1/\lambda} \\
&\le C_{d,p,q,\lambda}\left\|\left(\sum_{j\ge 0}\sum_{[\ell]\preceq 2^j}|\det A|^{j\alpha q}\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f\right|^{q}\right)^{\lambda/q}\right\|_{L^{p/\lambda}}^{1/\lambda} \\
&= C_{d,p,q,\lambda}\left\|\left(\sum_{j\ge 0}\sum_{[\ell]\preceq 2^j}[|\det A|^{j\alpha}\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f\right|]^{q}\right)^{1/q}\right\|_{L^p} \\
&= C_{d,p,q,\lambda}\,\|f\|_{\mathbf{F}^{\alpha,q}_p(AB)}\,.
\end{aligned}
$$

To prove the boundedness of T_ψ suppose $\mathbf{s}=\{s_Q\}_{Q\in\mathcal{Q}_{AB}}\in\mathbf{f}^{\alpha,q}_p$ and $f=T_\psi\mathbf{s}=\sum_{Q\in\mathcal{Q}_{AB}}s_Q\psi_Q$. Identify Q with (i,m,n). By Lemma 3.3 (see also Remark 3.4) and Lemma 3.1, we have for $x\in Q$ with $Q\in\mathcal{Q}^{i,m}$,

$$
\begin{aligned}
\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f(x)\right| &\le \sum_{i=j-1}^{j+1}\sum_{m(\ell,i)}\sum_{Q\in\mathcal{Q}^{i,m}}|s_Q|\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * \psi_Q(x)\right| \\
&\le C_N\sum_{i=j-1}^{j+1}\sum_{m(\ell,i)}\sum_{Q\in\mathcal{Q}^{i,m}}|s_Q|\,\frac{|Q|^{-1/2}}{(1+2^i\,|x-x_Q|)^N}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C'_N \sum_{i=j-1}^{j+1} \sum_{m(\ell,i)} \sum_{Q\in\mathcal{Q}^{i,m}} |s_Q| \frac{|Q|^{-1/2}}{(1+2^i|x_{Q'}-x_Q|)^N} \\
&= C'_N \sum_{i=j-1}^{j+1} \sum_{m(\ell,i)} |Q|^{-1/2} (s^*_{1,N})_Q \chi_Q(x) \\
&= C'_N \sum_{i=j-1}^{j+1} \sum_{m(\ell,i)} \sum_{Q\in\mathcal{Q}^{i,m}} (s^*_{1,N})_Q \tilde{\chi}_Q(x),
\end{aligned}
$$

for all $N > d$ and because $\mathcal{Q}^{i,m}$ is a partition of $\mathbb{R}^d$. Let $N > (d+1)\max(1, 1/q, 1/p)$. Then, (notice that, since $|i-j| \leq 1$, the change $Q \in \mathcal{Q}^{i,m}$ for $Q \in \mathcal{Q}^{j,\ell}$ is harmless) the previous estimate yields

$$
\begin{aligned}
\|T_\psi \mathbf{s}\|_{\mathbf{F}^{\alpha,q}_p(AB)} &\leq C_{d,p,q} \left\| \left(\sum_{j\geq 0} \sum_{[\ell]\preceq 2^j} \left[\sum_{Q\in\mathcal{Q}^{j,\ell}} |Q|^{-\alpha} (s^*_{1,N})_Q \tilde{\chi}_Q \right]^q \right)^{1/q} \right\|_{L^p} \\
&= C_{d,p,q} \left\| \left(\sum_{j\geq 0} \sum_{[\ell]\preceq 2^j} \sum_{Q\in\mathcal{Q}^{j,\ell}} [|Q|^{-\alpha} (s^*_{1,N})_Q \tilde{\chi}_Q]^q \right)^{1/q} \right\|_{L^p} \\
&= C_{d,p,q} \left\| \mathbf{s}^*_{1,N} \right\|_{\mathbf{f}^{\alpha,q}_p(AB)} \leq C_{d,p,q} \|\mathbf{s}\|_{\mathbf{f}^{\alpha,q}_p(AB)},
\end{aligned}
$$

because $\mathcal{Q}^{j,\ell}$ is a partition of $\mathbb{R}^d$ and Lemma 5.4 in the last inequality. ■

Remark 5.6. *With the same arguments as in Remark 2.6 in* [16]*, the definition of* $\mathbf{F}^{\alpha,q}_p(AB)$ *is independent of the choice of* $\psi \in \mathcal{S}$ *as long as it satisfies the requirements in Section 2.*

6. A reproducing identity with smooth Parseval frames

Recently, Guo and Labate in [22] found a way to overcome the use of characteristic functions in the Fourier domain to restrict the shearlets to the respective cone (see (2.7)). The use of these characteristic functions affects the smoothness of the *boundary shearlets* (those with $|[\ell]| = \pm 2^j$) in the Fourier domain and, therefore, their spatial localization. They slightly modify the definition of these boundary shearlets instead of projecting them into the cone. This new shearlets system is not affine-like. However, they do produce the same frequency covering as that in Section 2.

6.1. **The new smooth shearlets system.** This subsection is a brief summary of some results in [22] and is intended to show the construction of such smooth Parseval frames. Let $\hat{\phi}$ be a C^∞ univariate function such that $0 \leq \hat{\phi} \leq 1$, with $\hat{\phi} = 1$ on $[-1/16, 1/16]$ and $\hat{\phi} = 0$ outside $[-1/8, 1/8]$ (*i.e.*, ϕ is a rescaled scaling function of a Meyer wavelet). For $\xi \in \hat{\mathbb{R}}^d$, let $\hat{\Phi}(\xi) = \hat{\phi}(\xi_1)\hat{\phi}(\xi_2)\cdots\hat{\phi}(\xi_d)$ and $W^2(\xi) = \hat{\Phi}^2(2^{-2}\xi) -$

$\hat{\Phi}^2(\xi)$. It follows that

$$\hat{\Phi}(\xi) + \sum_{j\geq 0} W^2(2^{-2j}\xi) = 1, \quad \text{for all } \xi \in \hat{\mathbb{R}}^d.$$

Let now $v \in C^\infty(\mathbb{R})$ be such that $v(0) = 1$, $v^{(n)}(0) = 0$ for all $n \geq 1$, supp $v \subset [-1, 1]$ and

$$|v(u-1)|^2 + |v(u)|^2 + |v(u+1)|^2 = 1, \ \ |u| \leq 1.$$

Then, for any $j \geq 0$,

$$\sum_{m=-2^j}^{2^j} \left|v(2^j u - m)\right|^2 = 1, \ \ |u| \leq 1.$$

See [22] for comments on the construction of these functions and their properties. With $V_\mathfrak{d}(\xi) = \prod_{i\neq\mathfrak{d}} v(\frac{\xi_i}{\xi_\mathfrak{d}})$, $\xi \in \mathcal{D}^{(\mathfrak{d})}$, the shearlets system for $L^2((\mathcal{D}^{(\mathfrak{d})})^\vee)$ is defined as the countable collection of functions

$$\{\psi^{(\mathfrak{d})}_{j,\ell,k} : \mathfrak{d} = 1, \dots, d, j \geq 0, |[\ell]| \preceq 2^j, k \in \mathbb{Z}^d\},$$

whose "inner" elements ($|[\ell]| \prec 2^j$) are defined by their Fourier transform

$$(\psi^{(\mathfrak{d})}_{j,\ell,k})^\wedge(\xi) = \left|\det A_{(\mathfrak{d})}\right|^{-j/2} W(2^{-2j}\xi) V_\mathfrak{d}(\xi A^{-j}_{(\mathfrak{d})} B^{[-\ell]}_{(\mathfrak{d})}) \mathrm{e}^{-2\pi i \xi A^{-j}_{(\mathfrak{d})} B^{[-\ell]}_{(\mathfrak{d})} k}, \quad \xi \in \mathcal{D}^{(\mathfrak{d})}. \tag{6.1}$$

The *boundary shearlets* are defined slightly different but share similar properties in both the time and Fourier domain. This new system is not affine-like since the function W is not shear-invariant. However, they generate the same covering of $\hat{\mathbb{R}}^d$. The new smooth Parseval frame condition is now written as (see Theorem 2.3 in [22])

$$\left|\hat{\Psi}(\xi)\right|^2 + \sum_{\mathfrak{d}=1}^{d} \sum_{j\geq 0} \sum_{|[\ell]|\prec 2^j} \left|\hat{\psi}^{(\mathfrak{d})}(\xi A^{-j}_{(\mathfrak{d})} B^{[-\ell]}_{(\mathfrak{d})})\right|^2 + \sum_{\mathfrak{d}=1}^{d} \sum_{j\geq 0} \sum_{|[\ell]|=\pm 2^j} \left|\hat{\psi}^{(\mathfrak{d})}(\xi A^{-j}_{(\mathfrak{d})} B^{[-\ell]}_{(\mathfrak{d})})\right|^2 = 1, \tag{6.2}$$

for all $\xi \in \hat{\mathbb{R}}^d$. Notice that now there do not exist characteristic functions as in (2.7).

6.2. **The reproducing identity on $\mathcal{S}'$.** Our goal is to show that, with the smooth Parseval frames of shearlets of Guo and Labate in [22], $T_\psi \circ S_\psi$ is the identity on $\mathcal{S}'$. First, we show that any $f \in \mathcal{S}'$ admits a kind of Littlewood-Paley decomposition with shear anisotropic dilations, for which we follow [5]. Then, we show the reproducing identity in $\mathcal{S}'$ following [17].

Lemma 6.1. *Let $\{\Psi(\cdot - k)\}_{k\in\mathbb{Z}^d} \cup \{\psi_{j,\ell,k} : j \geq 0, |[\ell]| \preceq 2^j, k \in \mathbb{Z}^d\}$ be the smooth shearlet system that verifies (6.2). Then, for any $f \in \mathcal{S}'$,*

$$\begin{aligned} f = f * \tilde{\Psi} * \Psi \ \ &+ \ \ \sum_{\mathfrak{d}=1}^{d} \sum_{j\geq 0} \sum_{|[\ell]|\prec 2^j} f * \tilde{\psi}^{(\mathfrak{d})}_{A^{-j}_{(\mathfrak{d})} B^{[-\ell]}_{(\mathfrak{d})}} * \psi^{(\mathfrak{d})}_{A^{-j}_{(\mathfrak{d})} B^{[-\ell]}_{(\mathfrak{d})}} \\ &+ \ \ \sum_{\mathfrak{d}=1}^{d} \sum_{j\geq 0} \sum_{|[\ell]|=2^j} f * \tilde{\psi}^{(\mathfrak{d})}_{A^{-j}_{(\mathfrak{d})} B^{[-\ell]}_{(\mathfrak{d})}} * \psi^{(\mathfrak{d})}_{A^{-j}_{(\mathfrak{d})} B^{[-\ell]}_{(\mathfrak{d})}}, \end{aligned}$$

with convergence in $\mathcal{S}'$.

Proof. One can see Peetre's discussion on pp. 52-54 in [30] regarding convergence. Since the Fourier transform $\mathcal{F}$ is an isomorphism of $\mathcal{S}'$, it suffices to show that

$$\hat{f}(\xi) = \hat{f}(\xi)\left|\hat{\Psi}(\xi)\right|^2 + \sum_{\mathfrak{d}=1}^{d}\sum_{j\geq 0}\sum_{|[\ell]|\prec 2^j} \hat{f}(\xi)\left|\hat{\psi}^{(\mathfrak{d})}(\xi A_{(\mathfrak{d})}^{-j}B_{(\mathfrak{d})}^{-\ell})\right|^2 + \sum_{\mathfrak{d}=1}^{d}\sum_{j\geq 0}\sum_{|[\ell]|=\pm 2^j} \hat{f}(\xi)\left|\hat{\psi}^{(\mathfrak{d})}(\xi A_{(\mathfrak{d})}^{-j}B_{(\mathfrak{d})}^{-\ell})\right|^2$$

converges in $\mathcal{S}'$. Since the equality is a straight consequence of (6.2), we will only show convergence in $\mathcal{S}'$ of the right-hand side of the equality for those shearlets with $j \geq 0$ (Ψ is in fact a scaling function of a Meyer wavelet). Suppose that $\hat{f}$ has order $\leq m$. This is, there exists an integer $n \geq 0$ and a constant C such that

$$\left|\langle \hat{f}, g\rangle\right| \leq C \sup_{|\alpha|\leq n, |\beta|\leq m} \|g\|_{\alpha,\beta}, \quad \text{for all } g \in \mathcal{S},$$

where $\|g\|_{\alpha,\beta} = \sup_{\xi\in\hat{\mathbb{R}}^d} |\xi^\alpha| \left|\partial^\beta g(\xi)\right|$ denotes the usual semi-norm in $\mathcal{S}$ for multi-indices α and β. Then,

$$\left|\langle \hat{f}\left|(\psi_{A^{-j}B^{-\ell}})^\wedge\right|^2, g\rangle\right| = \left|\langle \hat{f}, \left|(\psi_{A^{-j}B^{-\ell}})^\wedge\right|^2 g\rangle\right| \leq C \sup_{|\alpha|\leq n, |\beta|\leq m} \left\|\left|(\psi_{A^{-j}B^{-\ell}})^\wedge\right|^2 g\right\|_{\alpha,\beta}.$$

As in Lemma 2.5 in [20], one can prove that

$$\sup_{|\beta|=m} \left\|\partial^\beta \left|(\psi_{A^{-j}B^{-\ell}})^\wedge\right|^2\right\|_\infty \leq C2^{-jm}.$$

Hence, by the compact support conditions of $(\psi_{A^{-j}B^{-\ell}})^\wedge(\xi)$ (see Section 1)

$$\begin{aligned}
&\sup_{|\alpha|\leq n, |\beta|\leq m} \left\|\left|(\psi_{A^{-j}B^{-\ell}})^\wedge\right|^2 g\right\|_{\alpha,\beta} \\
&\leq C \sup_{\xi\in\hat{\mathbb{R}}^2}\left[(1+|\xi|)^n \sup_{|\beta|\leq m}\left|\partial^\beta \left|(\psi_{A^{-j}B^{-\ell}})^\wedge(\xi)\right|^2\right| \sup_{|\beta|\leq m}\left|\partial^\beta g(\xi)\right|\right] \\
&\leq C \sup_{\xi\in\operatorname{supp}(\psi_{A^{-j}B^{-\ell}})^\wedge(\xi)} (1+|\xi|)^n \sup_{|\beta|\leq m}\left|\partial^\beta g(\xi)\right| \\
&\leq C \sup_{|\alpha|\leq n+1, |\beta|\leq m} \|g\|_{\alpha,\beta} \sup_{\xi\in\operatorname{supp}(\psi_{A^{-j}B^{-\ell}})^\wedge(\xi)} (1+|\xi|)^{-1} \\
&\leq C \sup_{|\alpha|\leq n+1, |\beta|\leq m} \|g\|_{\alpha,\beta} (1+2^{2j-4})^{-1} \leq C2^{-2j},
\end{aligned}$$

which proves the convergence in $\mathcal{S}'$. ∎

Theorem 6.2. *Let the shearlet system $\{\psi_{j,\ell,k}\}$ be constructed as in Subsection 6.1 such that it is a smooth Parseval frame that verifies (6.2). The composition of the analysis and synthesis operators $T_\psi \circ S_\psi$ (see (2.11) and (2.12) for the definitions) is the identity*

$$f = \sum_{Q\in\mathcal{Q}_{AB}} \langle f, \psi_Q\rangle \psi_Q,$$

in $\mathcal{S}'$.

Proof. As in (2.10), $f*\tilde{\psi}_{A^{-j}B^{-\ell}}(A^{-j}B^{-\ell}k) = f*\tilde{\psi}_{A^{-j}B^{-\ell}}(x_P) = |\det A|^{j/2}\langle f, \psi_P\rangle = \left|P_j^{-1/2}\right|\langle f, \psi_P\rangle$, where P is identified with (j, ℓ, k). Let $g = f * \tilde{\psi}_{A^{-j}B^{-\ell}}$ and $h = \psi_{A^{-j}B^{-\ell}}$. By construction, $\mathrm{supp}(\psi_{j,\ell,k})^\wedge(\xi) \subset QB^\ell A^j$. Therefore, Lemma 3.7 yields

$$\begin{aligned} f * \tilde{\psi}_{A^{-j}B^{-\ell}} * \psi_{A^{-j}B^{-\ell}} &= \sum_{k\in\mathbb{Z}^d} \langle f, \psi_{j,\ell,k}\rangle \psi_{j,\ell,k} \\ &= \sum_{P\in\mathcal{Q}^{j,\ell}} \langle f, \psi_P\rangle \psi_P. \end{aligned}$$

By appropriately summing over $\mathfrak{d} = 1, \ldots, d$, $j \geq 0$ and $|[\ell]| \preceq 2^j$, Lemma 6.1 yields the result. ∎

7. Embeddings

7.1. **Sobolev-type embeddings.** To prove the next embeddings we follow Section 2.3.2 in [31].

Theorem 7.1. *Let $s \in \mathbb{R}$.*

i) *For $0 < q_1 \leq q_0 \leq \infty$,*

$$\mathbf{B}_p^{s,q_1}(AB) \hookrightarrow \mathbf{B}_p^{s,q_0}(AB), \quad 0 < p \leq \infty$$

and

$$\mathbf{F}_p^{s,q_1}(AB) \hookrightarrow \mathbf{F}_p^{s,q_0}(AB), \quad 0 < p < \infty.$$

ii) *For $0 < q_0 \leq \infty$, $0 < q_1 \leq \infty$ and $\varepsilon > \frac{d-1}{(d+1)q_0}$,*

$$\mathbf{B}_p^{s+\varepsilon,q_1}(AB) \hookrightarrow \mathbf{B}_p^{s,q_0}(AB), \quad 0 < p \leq \infty$$

and

$$\mathbf{F}_p^{s+\varepsilon,q_1}(AB) \hookrightarrow \mathbf{F}_p^{s,q_0}(AB), \quad 0 < p < \infty.$$

iii) *For $0 < q \leq \infty$, $0 < p < \infty$ and $s \in \mathbb{R}$,*

$$\mathbf{B}_p^{s,\min\{p,q\}}(AB) \hookrightarrow \mathbf{F}_p^{s,q}(AB) \hookrightarrow \mathbf{B}_p^{s,\max\{p,q\}}(AB).$$

Proof. The monotonicity of ℓ^q norms proves i). To prove ii) let $\varepsilon > \frac{d-1}{(d+1)q_0}$. Then, the Besov case follows from

$$\begin{aligned} \|f\|_{\mathbf{B}_p^{s,q_0}} &\leq \sup_{\mathfrak{d},j,[\ell]} |Q_j|^{-(s+\varepsilon)} \left\| f * \psi^{(\mathfrak{d})}_{A^{-j}B^{-\ell}} \right\|_{L^p} \left(\sum_{\mathfrak{d}',j',[\ell']} |Q_j|^{\varepsilon q_0} \right)^{\frac{1}{q_0}} \\ &\lesssim \sup_{\mathfrak{d},j,[\ell]} |Q_j|^{-(s+\varepsilon)} \left\| f * \psi^{(\mathfrak{d})}_{A^{-j}B^{-\ell}} \right\|_{L^p} \left(\sum_{j'\geq 0} 2^{-j((d+1)\varepsilon q_0 - (d-1))} \right)^{\frac{1}{q_0}} \\ &= C_{\varepsilon,q_0,d} \sup_{\mathfrak{d},j,[\ell]} |Q_j|^{-(s+\varepsilon)} \left\| f * \psi^{(\mathfrak{d})}_{A^{-j}B^{-\ell}} \right\|_{L^p}, \end{aligned}$$

and $\ell^\infty \hookrightarrow \ell^{q_1}$. Similarly for the Triebel-Lizorkin case in ii). To prove iii) write $a_{j,\ell}(x) = |Q_j|^{-\alpha} f * \psi_{A^{-j}B^{-\ell}}(x)$. Since the set $\{j \geq 0, [|\ell|] \preceq 2^j\}$ is countable, one can find a bijection with $k \in \mathbb{Z}$. Consider first $0 < q \leq p < \infty$. Then,

$$\begin{aligned}\left(\sum_k \|a_k\|_{L^p}^p\right)^{\frac{1}{p}} &\leq \left(\int_{\mathbb{R}^n} (\sum_k |a_k(x)|^q)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \\ &\leq \left(\sum_k \||a_k(x)|^q\|_{L^{p/q}}\right)^{\frac{1}{q}} = \left(\sum_k \|a_k\|_{L^p}^q\right)^{\frac{1}{q}},\end{aligned}$$

because of usual ℓ^p inequalities and Minkowski's inequality. Now let $0 < p < q \leq \infty$. Then,

$$\begin{aligned}\left(\sum_k \|a_k\|_{L^p}^q\right)^{\frac{1}{q}} &= \left\|\int |a_k(x)|^p \, dx\right\|_{\ell^{q/p}}^{\frac{1}{p}} \leq \left(\int \||a_k(x)|^p\|_{\ell^{q/p}} \, dx\right)^{\frac{1}{p}} \\ &= \left(\int \||a_k(x)|\|_{\ell^q}^p \, dx\right)^{\frac{1}{p}} \leq \left(\int \|a_k(x)\|_{\ell^p}^p \, dx\right)^{\frac{1}{p}},\end{aligned}$$

because of the generalized Minkowski's inequality and usual ℓ^p inequalities.

■

7.2. **Embeddings of Besov spaces.** We now present embeddings between $\mathbf{B}_p^{\alpha_1,q}$ and $\mathbf{B}_p^{\alpha_2,q}(AB)$, for certain conditions on the smoothness parameters α_1 and α_2.

Theorem 7.2. *Let $0 < p, q \leq \infty$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. For $0 < p \leq 1$ let $\lambda = -1$ and for $1 < p < \infty$ let $\lambda > p/2$ ¿CASO $p = \infty$?. Then,*

$$\mathbf{B}_p^{\alpha_1,q} \hookrightarrow \mathbf{B}_p^{\alpha_2,q}(AB),$$

when $\frac{2d\lambda}{p} + \frac{d-1}{q} + (d+1)\alpha_2 < 2\alpha_1$.

Proof. To shorten notation write $\mathbf{b}_1 = \mathbf{b}_p^{\alpha_1,q}$, $\mathbf{B}_1 = \mathbf{B}_p^{\alpha_1,q}$ and $\mathbf{B}_2 = \mathbf{B}_p^{\alpha_2,q}(AB)$. Let $f = \sum_{Q \in \mathcal{D}_+} s_Q \varphi_Q \in \mathbf{B}_1$. From the compact support conditions on $(\varphi_{\nu,k})^\wedge$ and $(\psi_{A^{-j}B^{-\ell}})^\wedge$, we have

$$\begin{aligned}|f * \psi_{A^{-j}B^{-\ell}}(x)|^p &= \left|\sum_{\nu \geq 0} \sum_{k \in \mathbb{Z}^d} s_{\nu,k} \varphi_{\nu,k} * \psi_{A^{-j}B^{-\ell}}(x)\right|^p \\ &\lesssim \left(\sum_{k \in \mathbb{Z}^d} |s_{2j,k}| \, |\varphi_{2j,k} * \psi_{A^{-j}B^{-\ell}}(x)|\right)^p \\ &\leq C_N \left(\sum_{k \in \mathbb{Z}^d} |s_{2j,k}| \frac{|Q_{2j}|^{-1/2}}{(1 + 2^j |x + 2^{-2j}k|)^N}\right)^p,\end{aligned}$$

for every $N > d$ by Lemma 3.2. If $0 < p \leq 1$, choose $N > d/p$ and use the p-triangle inequality $|a+b|^p \leq |a|^p + |b|^p$ to get

$$\|f * \psi_{A^{-j}B^{-\ell}}\|_{L^p} \leq C_N \left(\sum_{k \in \mathbb{Z}^d} |s_{2j,k}|^p \, |Q_{2j}|^{-\frac{p}{2}} \int_{\mathbb{R}^d} \frac{dx}{(1 + 2^j |x + 2^{-2j}k|)^{Np}}\right)^{1/p}$$

$$\leq \quad C_N \left(\sum_{k\in\mathbb{Z}^d} |s_{2j,k}|^p \, |Q_{2j}|^{p(-\frac{1}{2}+\frac{1}{2p})} \right)^{1/p} .$$

For $1 < p < \infty$ choose $N = a + b$ such that $a > d/p$ and $b > d(p-1)/p$. Hölder's inequality yields

$$\begin{aligned}
|f * \psi_{A^{-j}B^{-\ell}}(x)|^p &\leq C_N \left(\sum_{k\in\mathbb{Z}^d} \frac{|s_{2j,k}| \, |Q_{2j}|^{-\frac{1}{2}}}{(1+2^j \, |x+2^{-2j}k|)^N} \cdot \frac{2^{jN}}{2^{jN}} \right)^p \\
&\leq C_N 2^{jNp} \left(\sum_{k\in\mathbb{Z}^d} \frac{|s_{2j,k}|^p \, |Q_{2j}|^{-\frac{p}{2}}}{(1+|2^{2j}x+k|)^{ap}} \right) \left(\sum_{k\in\mathbb{Z}^d} \frac{1}{(1+|2^{2j}x+k|)^{bp'}} \right)^{p/p'} \\
&\leq C_{N,p} 2^{jNp} \left(\sum_{k\in\mathbb{Z}^d} \frac{|s_{2j,k}|^p \, |Q_{2j}|^{-\frac{p}{2}}}{(1+|2^{2j}x+k|)^{ap}} \right).
\end{aligned}$$

With $\lambda = \frac{Np}{2d} > p/2$, we write $2^{jNp} = |Q_{2j}|^{-\frac{pN}{2d}} = |Q_{2j}|^{-\lambda}$. Since $ap > d$ we have

$$\begin{aligned}
\|f * \psi_{A^{-j}B^{-\ell}}\|_{L^p} &\leq C_{N,p} \left(\int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}^d} \frac{|s_{2j,k}|^p \, |Q_{2j}|^{p(-\frac{1}{2}-\frac{\lambda}{p})}}{(1+|2^{2j}x+k|)^{ap}} dx \right)^{1/p} \\
&= C_{N,p} \left(\sum_{k\in\mathbb{Z}^d} |s_{2j,k}|^p \, |Q_{2j}|^{p(-\frac{1}{2}-\frac{\lambda}{p}+\frac{1}{p})} \right)^{1/p} .
\end{aligned}$$

Let $\lambda = -1$ for $0 < p \leq 1$ and $\lambda > p/2$ for $1 < p < \infty$. Then, since $|\{\ell : |[\ell]| \leq 2^j\}| \sim 2^{(d-1)j} = |Q_{2j}|^{-\frac{d-1}{2d}}$ and $|P_j|^{-\alpha_2} = 2^{j(d+1)\alpha_2} = |Q_{2j}|^{-\frac{(d+1)\alpha_2}{2d}}$, we have

$$\begin{aligned}
\|f\|_{\mathbf{B}_2} &\leq C_{N,p} \left(\sum_{j\geq 0} \sum_{|[\ell]|\leq 2^j} \left[|P_j|^{-\alpha_2} \left(\sum_{k\in\mathbb{Z}^d} [|Q_{2j}|^{-\frac{1}{2}+\frac{1}{p}-\frac{\lambda}{p}} \, |s_{2j,k}|]^p \right)^{1/p} \right]^q \right)^{1/q} \\
&\leq C_{N,p} \left(\sum_{j\geq 0} \left(\sum_{k\in\mathbb{Z}^d} [|Q_{2j}|^{-\frac{1}{2}+\frac{1}{p}-\frac{\lambda}{p}-\frac{d-1}{2qd}-\frac{(d+1)\alpha_2}{2d}} \, |s_{2j,k}|]^p \right)^{q/p} \right)^{1/q} \\
&= C_{N,p} \left(\sum_{j\geq 0} \left(\sum_{Q\in\mathcal{D}^{2j}} [|Q|^{-\frac{1}{2}+\frac{1}{p}-\frac{\lambda}{p}-\frac{d-1}{2qd}-\frac{(d+1)\alpha_2}{2d}} \, |s_Q|]^p \right)^{q/p} \right)^{1/q} \\
&\leq C_{N,p} \left(\sum_{j\geq 0} \left(\sum_{Q\in\mathcal{D}^{2j}} [|Q|^{-\frac{\alpha_1}{d}+\frac{1}{p}-\frac{1}{2}} \, |s_Q|]^p \right)^{q/p} \right)^{1/q} \\
&\leq C_{N,p} \left(\sum_{j\geq 0} \left(\sum_{Q\in\mathcal{D}^{j}} [|Q|^{-\frac{\alpha_1}{d}+\frac{1}{p}-\frac{1}{2}} \, |s_Q|]^p \right)^{q/p} \right)^{1/q} = C_{N,p} \|\mathbf{s}\|_{\mathbf{b}_1} .
\end{aligned}$$

By Theorem 2.6 in [15] (with slight different notation for the analysis operator and, therefore, for the definition of the space), the last norm is bounded by $\|f\|_{\mathbf{B}_1}$. This completes the proof. ■

Theorem 7.3. *Let* $0 < p, q \leq \infty$, $s = [\max(1, 1/p) - \min(1, 1/q)]/2$ *and* $\alpha_1, \alpha_2 \in \mathbb{R}$. *Let* $\lambda = 0$ *if* $0 < p \leq 1$ *or* $\lambda > d(p-1)/p$ *if* $1 < p < \infty$ *and* $2d + \lambda + 2(\alpha_1 + s(d-1)) < (d+1)(\alpha_2 + 1)$. *Then,*

$$\mathbf{B}_p^{\alpha_2,q}(AB) \hookrightarrow \mathbf{B}_p^{\alpha_1,q}$$

Proof. Let $f = \sum_{P\in\mathcal{Q}_{AB}} s_P\psi_P \in \mathbf{B}_p^{\alpha_2,q}(AB)$. To shorten notation write $\mathbf{B}_1 = \mathbf{B}_p^{\alpha_1,q}$ and $\mathbf{B}_2 = \mathbf{B}_p^{\alpha_2,q}(AB)$. Let $p < 1$ and $q \geq 1$. Since $\|\cdot\|_{L^p}^p$ satisfies the triangular inequality, using Hölder's inequality $(1/p > 1)$ we have that

$$\begin{aligned}
\|f\|_{\mathbf{B}_1} &\lesssim \left(\sum_{j=0}^{\infty} |Q_{2j}|^{-\frac{\alpha_1 q}{d}} \left[\sum_{|[\ell]|\preceq 2^j} \left\|\sum_{P\in\mathcal{Q}^{j,\ell}} s_P\psi_P * \varphi_{2^{2j}I}\right\|_{L^p}^p\right]^{q/p}\right)^{1/q} \\
&\lesssim \left(\sum_{j=0}^{\infty} |Q_{2j}|^{-\frac{\alpha_1 q}{d}}\, 2^{(d-1)jq(\frac{1}{p}-1)} \left[\sum_{|[\ell]|\preceq 2^j} \left\|\sum_{P\in\mathcal{Q}^{j,\ell}} s_P\psi_P * \varphi_{2^{2j}I}\right\|_{L^p}\right]^{q}\right)^{1/q},
\end{aligned}$$

since $|\{\ell : |[\ell]| \leq 2^j\}| \sim 2^{(d-1)j}$. Using Hölder's inequality again $(q \geq 1)$,

$$\|f\|_{\mathbf{B}_1} \lesssim \left(\sum_{j=0}^{\infty} |Q_{2j}|^{-\frac{\alpha_1 q}{d}}\, 2^{2j(d-1)q(\frac{1}{p}-\frac{1}{q})/2} \sum_{|[\ell]|\preceq 2^j} \left\|\sum_{P\in\mathcal{Q}^{j,\ell}} s_P\psi_P * \varphi_{2^{2j}I}\right\|_{L^p}^q\right)^{1/q}.$$

The other cases are result of the triangular inequality when $p \geq 1$ and the fact that $\ell^q \hookrightarrow \ell^1$ when $q < 1$. Since $\psi_P(x) = |P_j|^{-\frac{1}{2}}\,\psi(B^{[\ell]}A^j x - k)$ and $\varphi_{2^{2j}I}(x) = |Q_{2j}|^{-1}\,\varphi(2^{2j}x)$, Lemma 3.2 yields

$$\begin{aligned}
&\left\|\sum_{P\in\mathcal{Q}^{j,\ell}} s_P\psi_P * \varphi_{2^{2j}I}\right\|_{L^p}^q \\
&\quad\leq \left(\int_{\mathbb{R}^d}\left[\sum_{P\in\mathcal{Q}^{j,\ell}} |s_P|\,|\psi_P * \varphi_{2^{2j}I}(x)|\right]^p dx\right)^{q/p} \\
&\quad\leq \left(\int_{\mathbb{R}^d}\left[\sum_{k\in\mathbb{Z}^d} |s_{j,\ell,k}|\,\frac{|P_j|^{-\frac{1}{2}}\,|Q_{2j}|^{-1}\,|P_j|}{(1 + 2^j\,|x - A^{-j}B^{-[\ell]}k|)^N}\right]^p dx\right)^{q/p},
\end{aligned}$$

for every $N > d$. When $0 < p \leq 1$ choose $Np > d$. Then, the p-triangular inequality $|a+b|^p \leq |a|^p + |b|^p$ yields

$$\begin{aligned}
\left\|\sum_{P\in\mathcal{Q}^{j,\ell}} s_P\varphi_{2^{2j}I} * \psi_P\right\|_{L^p}^q &\leq C_N\left(\sum_{k\in\mathbb{Z}^d} |s_{j,\ell,k}|^p\,|Q_{2j}|^{-p}\,|P_j|^{\frac{p}{2}}\,|Q_{2j}|^{\frac{1}{2}}\right)^{q/p} \\
&= C_N\left(\sum_{k\in\mathbb{Z}^d} |s_{j,\ell,k}|^p\,|P_j|^{\frac{p}{2}}\,|P_j|^{-\frac{2dp}{d+1}}\,|P_j|^{\frac{d}{d+1}}\right)^{q/p}.
\end{aligned}$$

If $1 < p < \infty$ choose $N = a + b$ such that $a > d/p$ and $b > d(p-1)/p$. Hölder's inequality gives

$$\left(\sum_{k\in\mathbb{Z}^d} \frac{|s_{j,\ell,k}|\,|Q_{2j}|^{-1}\,|P_j|^{\frac{1}{2}}}{(1+2^j\,|x-A^{-j}B^{-[\ell]}k|)^N}\right)^p$$
$$\leq \left(\sum_{k\in\mathbb{Z}^d} \frac{|s_{j,\ell,k}|^p\,|Q_{2j}|^{-p}\,|P_j|^{\frac{p}{2}}}{(1+2^j\,|x-A^{-j}B^{-[\ell]}k|)^{ap}}\right)\left(\sum_{k\in\mathbb{Z}^d} \frac{1}{(1+2^j\,|x-A^{-j}B^{-[\ell]}k|)^{bp'}}\cdot\frac{2^{jbp'}}{2^{jbp'}}\right)^{p/p'}$$
$$\leq C_{d,p}2^{jbp}\left(\sum_{k\in\mathbb{Z}^d} \frac{|s_{j,\ell,k}|^p\,|Q_{2j}|^{-p}\,|P_j|^{\frac{p}{2}}}{(1+2^j\,|x-A^{-j}B^{-[\ell]}k|)^{ap}}\right),$$

because $\left|A^{-j}B^{-[\ell]}x\right| \geq 2^{-2(j-1)}\,|x|$ by Lemma 8.1. Since $2^{jbp} = |P_j|^{-\frac{bp}{d+1}}$ and $|Q_{2j}|^{-p} = |P_j|^{-\frac{2dp}{d+1}}$, we have

$$\begin{aligned}\left\|\sum_{P\in\mathcal{Q}^{j,\ell}} s_P\varphi_{2^{2j}I}*\psi_P\right\|_{L^p}^q &\leq C_{d,p}\left(\sum_{k\in\mathbb{Z}^d}|s_{j,\ell,k}|^p\,|P_j|^{-\frac{bp}{d+1}-\frac{2dp}{d+1}+\frac{p}{2}}\,|Q_{2j}|^{\frac{1}{2}}\right)^{q/p}\\ &= C_{d,p}\left(\sum_{k\in\mathbb{Z}^d}|s_{j,\ell,k}|^p\,|P_j|^{-\frac{bp}{d+1}-\frac{2dp}{d+1}+\frac{p}{2}+\frac{d}{d+1}}\right)^{q/p}.\end{aligned}$$

Let $\lambda = 0$ if $0 < p \leq 1$ or $\lambda > d(p-1)/p$ if $1 < p < \infty$. Then,

$$\left\|\sum_{P\in\mathcal{Q}^{j,\ell}} s_P\varphi_{2^{2j}I}*\psi_P\right\|_{L^p}^q \leq C_{d,p}\left(\sum_{k\in\mathbb{Z}^d}[|s_{j,\ell,k}|\,|P_j|^{\frac{1}{2}-\frac{2d+\lambda}{d+1}+\frac{d}{p(d+1)}}]^p\right)^{q/p}.$$

Finally,

$$\begin{aligned}\|f\|_{\mathbf{B}_1} &\leq C_{d,p}\left(\sum_{j=0}^{\infty}|Q_{2j}|^{-\frac{\alpha_1 q}{d}}\,|Q_{2j}|^{-\frac{qs(d-1)}{d}}\sum_{|[\ell]|\preceq 2^j}\left\|\sum_{P\in\mathcal{Q}^{j,\ell}} s_P\varphi_{2^{2j}I}*\psi_P\right\|_{L^p}^q\right)^{1/q}\\ &\leq C_{d,p}\left(\sum_{j=0}^{\infty}|P_j|^{-\frac{2\alpha_1 q}{d+1}-\frac{2qs(d-1)}{d+1}}\sum_{|[\ell]|\preceq 2^j}\left(\sum_{k\in\mathbb{Z}^d}|s_{j,\ell,k}|^p\,|P_j|^{p(\frac{1}{2}-\frac{2d+\lambda}{d+1}+\frac{d}{p(d+1)})}\right)^{q/p}\right)^{1/q}\\ &= C_{d,p}\left(\sum_{j=0}^{\infty}\sum_{|[\ell]|\preceq 2^j}\left(\sum_{k\in\mathbb{Z}^d}[|s_{j,\ell,k}|\,|P_j|^{\frac{1}{2}-\frac{2d+\lambda}{d+1}+\frac{d}{p(d+1)}-\frac{2(\alpha_1+s(d-1))}{d+1}}]^p\right)^{q/p}\right)^{1/q}\\ &\leq C_{d,p}\left(\sum_{j=0}^{\infty}\sum_{|[\ell]|\preceq 2^j}\left(\sum_{k\in\mathbb{Z}^d}[|s_{j,\ell,k}|\,|P_j|^{-\alpha_2+\frac{d}{p(d+1)}-\frac{1}{2}}]^p\right)^{q/p}\right)^{1/q} = C_{d,p}\,\|\mathbf{s}\|_{\mathbf{b}_p^{\alpha_2,q}},\end{aligned}$$

because $2d + \lambda + 2(\alpha_1 + s(d-1)) < (d+1)(\alpha_2 + 1)$. Applying Theorem 4.3 finishes the proof. ■

7.3. **Vanishing norms of non-vanishing functions on Besov spaces.** We now show that there exist sequences of non vanishing functions in the norm of any of the shear anisotropic or isotropic spaces that vanish in the norm of the other space.

Theorem 7.4. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $0 < p_1, p_2, q_1, q_2 \leq \infty$ be such that $2\alpha_1 + \frac{3}{2}d < (d+1)(\alpha_2 - \frac{d}{p_2(d+1)} + \frac{1}{2}) + \frac{1}{2} + \frac{d}{p_1}$. Then, there are sequences of functions in $\mathbf{B}_{p_2}^{\alpha_2,q_2}(AB)$, with $\|f^{(j)}\|_{\mathbf{B}_{p_2}^{\alpha_2,q_2}(AB)} \approx 1$, for all $j \in \mathbb{N}$, but $\lim_{j\to\infty} \|f^{(j)}\|_{\mathbf{B}_{p_1}^{\alpha_1,q_1}} \to 0$.*

Proof. For $j \geq 0$, let $P_j = P_{j,0,0} \in \mathcal{Q}_{AB}$ and define $\mathbf{s}^{(j)} = \{s_Q^{(j)}\}_{Q\in\mathcal{Q}_{AB}}$ such that

$$s_Q^{(j)} = \left\{ \begin{array}{cc} 0 & \text{if} \quad Q \neq P_j \\ |P_j|^{\alpha_2 - \frac{d}{p_2(d+1)} + \frac{1}{2}} & \text{if} \quad Q = P_j. \end{array} \right\}$$

Then, $\|\mathbf{s}^{(j)}\|_{\mathbf{b}_{p_2}^{\alpha_2,q_2}(AB)} = 1$, for all $j \geq 0$. Thus, $f^{(j)}(x) = \sum_{Q\in\mathcal{Q}_{AB}} s_Q^{(j)} \psi_Q(x) = |P_j|^{\alpha_2 - \frac{d}{p_2(d+1)} + \frac{1}{2}} \psi_{j,0,0} \in \mathbf{F}_{p_2}^{\alpha_2,q_2}(AB)$ with $\|f\|_{\mathbf{F}_{p_2}^{\alpha_2,q_2}(AB)} \approx 1$, for all $j \geq 0$. From the support conditions on $\hat{\varphi}$ and $\hat{\psi}$, we have

$$\|f^{(j)}\|_{\mathbf{B}_{p_1}^{\alpha_1,q_1}} = \left(\sum_{\nu=(2j-5)_+}^{(2j-1)_+} (|Q_\nu|^{-\frac{\alpha_1}{d}} |P_j|^{\alpha_2 - \frac{d}{p_2(d+1)} + \frac{1}{2}} \|\varphi_{2^\nu I} * \psi_{j,0,0}\|_{L^{p_1}})^{q_1} \right)^{1/q_1}.$$

Assume $\nu \sim 2j$. Since $\varphi_{2^{2j}I}(x) = 2^{2jd}\varphi(2^{2j}x) = |Q_{2j}|^{-1} \varphi(2^{2j}x)$ and $\psi_{j,0,0}(x) = |\det A|^{\frac{j}{2}} \psi(A^j x) = |P_j|^{-\frac{1}{2}} \psi(A^j x)$, Lemma 3.2 yields

$$|\varphi_{2^{2j}I} * \psi_{j,0,0}(x)| \leq \frac{C_N |Q_{2j}|^{-1} |P_j|^{\frac{1}{2}}}{(1 + 2^j |x|)^N},$$

for some $C_N > 0$ for all $N > d$. Taking $N > \max\{d, d/p_1\}$, $\|\varphi_{2^{2j}I} * \psi_{j,0,0}(x)\|_{L^{p_1}} \leq C_{d,p_1} |Q_{2j}|^{-1} |P_j|^{\frac{1}{2}} |Q_{2j}|^{\frac{1}{2p_1}}$. Hence,

$$\begin{aligned} \|f^{(j)}\|_{\mathbf{B}_{p_1}^{\alpha_1,q_1}} &\lesssim C_{d,p_1} |Q_{2j}|^{-\frac{\alpha_1}{d}} |P_j|^{\alpha_2 - \frac{d}{p_2(d+1)} + \frac{1}{2}} \cdot |P_j|^{\frac{1}{2}} |Q_{2j}|^{-1+\frac{1}{2p_1}} \\ &= 2^{-j(-2\alpha_1 + (d+1)(\alpha_2 - \frac{d}{p_2(d+1)} + \frac{1}{2}) + \frac{d+1}{2} + 2d(-1+\frac{1}{2p_1}))}, \end{aligned}$$

tends to 0, as $j \to \infty$, because $2\alpha_1 + \frac{3}{2}d < (d+1)(\alpha_2 - \frac{d}{p_2(d+1)} + \frac{1}{2}) + \frac{1}{2} + \frac{d}{p_1}$. ■

Theorem 7.5. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $0 < p_1, p_2, q_1, q_2 \leq \infty$ be such that $2\alpha_1 + d > (d+1)\alpha_2 + \frac{d-1}{q_2} + \frac{2d}{p_1}$. Then, there are sequences of functions in $\mathbf{B}_{p_1}^{\alpha_1,q_1}$, with $\|f^{(\nu)}\|_{\mathbf{B}_{p_1}^{\alpha_1,q_1}} \approx 1$, for all $\nu \in \mathbb{N}$, but $\lim_{\nu\to\infty} \|f^{(\nu)}\|_{\mathbf{B}_{p_2}^{\alpha_2,q_2}(AB)} \to 0$.*

Proof. For $\nu \geq 0$, let $Q_\nu = Q_{\nu,0} \in \mathcal{D}_+$ and define $\mathbf{s}^{(\nu)} = \{s_Q^{(\nu)}\}_{Q\in\mathcal{D}_+}$ such that

$$s_Q^{(\nu)} = \left\{ \begin{array}{cc} 0 & \text{if} \quad Q \neq Q_\nu \\ |Q_\nu|^{\alpha_1 - \frac{1}{p_1} + \frac{1}{2}} & \text{if} \quad Q = Q_\nu. \end{array} \right\}$$

Then, $\|\mathbf{s}^{(\nu)}\|_{\mathbf{b}_{p_1}^{\alpha_1,q_1}} = 1$, for all $\nu \geq 0$. Thus, $f^{(\nu)}(x) = \sum_{Q\in\mathcal{D}_+} s_Q^{(\nu)} \varphi_Q(x) = |Q_\nu|^{\alpha_2 - \frac{1}{p_2} + \frac{1}{2}} \varphi_{\nu,0} \in \mathbf{F}_{p_1}^{\alpha_1,q_1}$ with $\|f\|_{\mathbf{F}_{p_1}^{\alpha_1,q_1}} \approx 1$, for all $\nu \geq 0$. Consider the subsequence $f^{(2\nu)}$. From

the compact support conditions on $\hat{\varphi}$ and $\hat{\psi}$, we have

$$\left\|f^{(2\nu)}\right\|_{\mathbf{B}_2} \lesssim \left(\sum_{|[\ell]|\preceq 2^\nu} [|P_\nu|^{-\alpha_2} |Q_{2\nu}|^{\frac{\alpha_1}{d}-\frac{1}{p_1}+\frac{1}{2}} \left\|\varphi_{2\nu,0} * \psi_{A^{-\nu}B^{-\ell}}\right\|_{L^{p_2}}]^{q_2}\right)^{1/q_2}.$$

Since $\varphi_{2\nu,0}(x) = |Q_{2\nu}|^{-\frac{1}{2}} \varphi(2^{2\nu}x)$ and $\psi_{A^{-\nu}B^{-\ell}}(x) = |P_\nu|^{-1} \psi(B^\ell A^\nu x)$, Lemma 3.2 yields

$$|\varphi_{2\nu,0} * \psi_{A^{-\nu}B^{-\ell}}(x)| \leq \frac{C_N |Q_{2\nu}|^{-\frac{1}{2}}}{(1+2^\nu|x|)^N},$$

for some $C_N > 0$ for all $N > d$ and all ℓ such that $|[\ell]| \preceq 2^\nu$. Taking $N > \max\{d, d/p_2\}$, $\left\|\varphi_{2\nu,0} * \psi_{A^{-\nu}B^{-\ell}}\right\|_{L^{p_2}} \leq C_{d,p_2}$. Therefore, since $|\{\ell : |[\ell]| \preceq 2^\nu\}| = 2^{(\nu+1)(d-1)} + 1 \leq C_d 2^{\nu(d-1)}$, we finally get

$$\begin{aligned}
\left\|f^{(2\nu)}\right\|_{\mathbf{B}_2} &\leq C_{d,p_2} \left(\sum_{|[\ell]|\preceq 2^\nu} [|P_\nu|^{-\alpha_2} |Q_{2\nu}|^{\frac{\alpha_1}{d}-\frac{1}{p_1}+\frac{1}{2}}]^{q_2}\right)^{1/q_2} \\
&\leq C_{d,p_2} \left([2^{\frac{\nu(d-1)}{q_2}} 2^{\nu(d+1)\alpha_2} 2^{-2\nu d(\frac{\alpha_1}{d}-\frac{1}{p_1}+\frac{1}{2})}]^{q_2}\right)^{1/q_2} \\
&\leq C_{d,p_2} 2^{-\nu[-\frac{d-1}{q_2}-(d+1)\alpha_2+2d(\frac{\alpha_1}{d}-\frac{1}{p_1}+\frac{1}{2})]},
\end{aligned}$$

which tends to 0, as $\nu \to \infty$, if $2\alpha_1 + d > (d+1)\alpha_2 + \frac{d-1}{q_2} + \frac{2d}{p_1}$. ■

7.4. **Embeddings of Triebel-Lizorkin spaces.**

Theorem 7.6. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$, $0 < q \leq \infty$, $0 < p < \infty$ and $\lambda > d\max(1, 1/q, 1/p)$. If $(d+1)\alpha_2 + \frac{d-1}{q} + \lambda \leq 2\alpha_1$,*

$$\mathbf{F}_p^{\alpha_1,q} \hookrightarrow \mathbf{F}_p^{\alpha_2,q}(AB).$$

Proof. We only prove for $q < \infty$, case $q = \infty$ is similar. To shorten notation write $\mathbf{F}_1 = \mathbf{F}_p^{\alpha_1,q}$, $\mathbf{f}_1 = \mathbf{f}_p^{\alpha_1,q}$ and $\mathbf{F}_2 = \mathbf{F}_p^{\alpha_2,q}(AB)$. From the compact support conditions on $\hat{\varphi}$ and $\hat{\psi}$, and since $\varphi_{2j,k}(x) = |Q_{2j}|^{-\frac{1}{2}} \varphi(2^{2j}x - k)$ and $\psi_{A^{-j}B^{-\ell}}(x) = |P_j|^{-1} \psi(B^\ell A^j x)$, Lemma 3.2 yields

$$|\varphi_{2j,k} * \psi_{A^{-j}B^{-\ell}}(x)| \leq \frac{C_N |Q_{2j}|^{-\frac{1}{2}}}{(1+2^j|x+2^{-2j}k|)^N},$$

for some $C_N > 0$ for all $N > d$ and all ℓ such that $|[\ell]| \preceq 2^j$. Then,

$$\|f\|_{\mathbf{F}_2} \lesssim C_N \left\|\left(\sum_{j\geq 0} |P_j|^{-\alpha_2 q} \sum_{|[\ell]|\preceq 2^j} [\sum_{k\in\mathbb{Z}^d} |s_{2j,k}| \frac{|Q_{2j}|^{-\frac{1}{2}}}{(1+2^j|\cdot+2^{-2j}k|)^N}]^q\right)^{1/q}\right\|_{L^p},$$

for all $N > d$. Let $\lambda > d\max(1, 1/q, 1/p)$. Following the proof of the second part of Theorem 5.5, if $x \in Q$ and $Q \in \mathcal{D}^{2j}$,

$$[\sum_{k\in\mathbb{Z}^d} \frac{|s_{2j,k}| |Q_{2j}|^{-\frac{1}{2}}}{(1+2^j|x-2^{-2j}k|)^\lambda}]^q$$

$$
\begin{aligned}
&= [\sum_{k\in\mathbb{Z}^d} \frac{|s_{2j,k}|\,|Q_{2j}|^{-\frac{1}{2}}\cdot 2^{j\lambda}}{2^{j\lambda}(1+2^j\,|x-2^{-2j}k|)^{\lambda}}]^q \le [2^{j\lambda}\sum_{k\in\mathbb{Z}^d} \frac{|s_{2j,k}|\,|Q_{2j}|^{-\frac{1}{2}}}{(1+2^{2j}\,|x-2^{-2j}k|)^{\lambda}}]^q \\
&\lesssim [2^{j\lambda}\sum_{Q\in\mathcal{D}^{2j}} |Q|^{-\frac{1}{2}}\left|(s^*_{1,\lambda})_Q\right|\chi_Q(x)]^q = 2^{j\lambda q}\sum_{Q\in\mathcal{D}^{2j}} [\left|(s^*_{1,\lambda})_Q\right|\tilde{\chi}_Q(x)]^q,
\end{aligned}
$$

because $\mathcal{D}^{2j}$ is a partition of $\mathbb{R}^d$. Since $|\{\ell : |[\ell]| \preceq 2^j\}| = 2^{(j+1)(d-1)} + 1 \le C_d 2^{j(d-1)}$, we have

$$
\begin{aligned}
\|f\|_{\mathbf{F}_2} &\lesssim C_{d,p,q}\left\|\left(\sum_{j\ge 0}|P_j|^{-\alpha_2 q}\,2^{j(d-1)}2^{j\lambda q}\sum_{Q\in\mathcal{D}^{2j}}[\left|(s^*_{1,\lambda})_Q\right|\tilde{\chi}_Q(\cdot)]^q\right)^{1/q}\right\|_{L^p} \\
&\le C_{d,p,q}\left\|\left(\sum_{j\ge 0}\sum_{Q\in\mathcal{D}^{2j}}[|Q|^{-\frac{\alpha_1}{d}}\left|(s^*_{1,\lambda})_Q\right|\tilde{\chi}_Q(\cdot)]^q\right)^{1/q}\right\|_{L^p} \\
&\le C_{d,p,q}\left\|\left(\sum_{j\ge 0}\sum_{Q\in\mathcal{D}^{j}}[|Q|^{-\frac{\alpha_1}{d}}\left|(s^*_{1,\lambda})_Q\right|\tilde{\chi}_Q(\cdot)]^q\right)^{1/q}\right\|_{L^p} = \left\|\mathbf{s}^*_{1,\lambda}\right\|_{\mathbf{f}_1},
\end{aligned}
$$

because $(d+1)j\alpha_2 + \frac{(d-1)j}{q} + j\lambda \le 2j\alpha_1$. Following the proof of Lemma 2.3 of [16] (where the restriction on λ is used) it can be concluded that $\left\|\mathbf{s}^*_{1,\lambda}\right\|_{\mathbf{f}_1} \lesssim \|\mathbf{s}\|_{\mathbf{f}_1}$, and from Theorem 2.2 in [16], $\|\mathbf{s}\|_{\mathbf{f}_1} \lesssim \|f\|_{\mathbf{F}_1}$, which finishes the proof. ■

Theorem 7.7. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$, $0 < q \le \infty$ and $0 < p < \infty$ be such that $2\alpha_1 + d + (d-1)(1-1/q)_+ \le (d+1)\alpha_2 + 1$, where $(1-1/q)_+ = \max\{0, 1-1/q\}$. Then,*

$$
\mathbf{F}^{\alpha_2,q}_p(AB) \hookrightarrow \mathbf{F}^{\alpha_1,q}_p.
$$

Proof. We only prove for $q < \infty$, case $q = \infty$ is similar. To shorten notation write $\mathbf{F}_1 = \mathbf{F}^{\alpha_1,q}_p$, $\mathbf{F}_2 = \mathbf{F}^{\alpha_2,q}_p(AB)$ and $\mathbf{f}_2 = \mathbf{f}^{\alpha_2,q}_p(AB)$. Suppose $f = \sum_{P\in\mathcal{Q}_{AB}} s_P\psi_P \in \mathbf{F}_2$. From the compact support conditions on $(\varphi_{2^\nu I})^\wedge$ and $(\psi_{j,\ell,k})^\wedge$, we have

$$
\begin{aligned}
\|f\|_{\mathbf{F}_1} &= \left\|\left(\sum_{\nu\ge 0}(|Q_\nu|^{-\frac{\alpha_1}{d}}\,|\varphi_{2^\nu I} * f(\cdot)|)^q\right)^{1/q}\right\|_{L^p} \\
&\lesssim \left\|\left(\sum_{\nu\ge 0}|Q_{2\nu}|^{-\frac{\alpha_1 q}{d}}(\sum_{|[\ell]|\preceq 2^\nu}\sum_{k\in\mathbb{Z}^d}|s_{\nu,\ell,k}|\,|\varphi_{2^{2\nu}I} * \psi_{\nu,\ell,k}(\cdot)|)^q\right)^{1/q}\right\|_{L^p} \\
&\lesssim C_N\left\|\left(\sum_{\nu\ge 0}|Q_{2\nu}|^{-\frac{\alpha_1 q}{d}}(\sum_{|[\ell]|\preceq 2^\nu}\sum_{k\in\mathbb{Z}^d}|s_{\nu,\ell,k}|\,\frac{|Q_{2\nu}|^{-1}\,|P_\nu|^{-1/2}\,|P_\nu|}{(1+2^\nu\,|\cdot - A^{-\nu}B^{-\ell}k|)^N})^q\right)^{1/q}\right\|_{L^p},
\end{aligned}
$$

for some $C_N > 0$ for all $N > d$, by Lemma 3.2. Continuing as in the second part of the proof of Theorem 5.5, if $x \in P$ and $P \in \mathcal{Q}^{\nu,\ell}$,

$$\begin{aligned}
\|f\|_{\mathbf{F}_1} &\lesssim C_N \left\| \left(\sum_{\nu\geq 0} |Q_{2\nu}|^{-\frac{\alpha_1 q}{d}-q} |P_\nu|^q [\sum_{|[\ell]|\preceq 2^\nu} \sum_{P\in\mathcal{Q}^{\nu,\ell}} |s_P| \cdot \frac{|P|^{-1/2}}{(1+2^\nu|\cdot - x_P|)^N}]^q \right)^{1/q} \right\|_{L^p} \\
&\lesssim C_N \left\| \left(\sum_{\nu\geq 0} |P_\nu|^{\frac{2dq}{d+1}(-\frac{\alpha_1}{d}-1)+q} [\sum_{|[\ell]|\preceq 2^\nu} \sum_{P\in\mathcal{Q}^{\nu,\ell}} (s^*_{1,N})_P \tilde{\chi}_P(\cdot)]^q \right)^{1/q} \right\|_{L^p},
\end{aligned}$$

because $\mathcal{Q}^{\nu,\ell}$ is a partition of $\mathbb{R}^d$. However, $\sum_{|[\ell]|\preceq 2^\nu} \sum_{P\in\mathcal{Q}^{\nu,\ell}} \chi_P$ is not a partition of $\mathbb{R}^d$. If $0 < q \leq 1$ we use the q-triangle inequality $|a+b|^q \leq |a|^q + |b|^q$ ($N > d/q$) to get

$$[\sum_{|[\ell]|\preceq 2^\nu} \sum_{P\in\mathcal{Q}^{\nu,\ell}} (s^*_{1,N})_P \tilde{\chi}_P(\cdot)]^q \leq \sum_{|[\ell]|\preceq 2^\nu} \sum_{P\in\mathcal{Q}^{\nu,\ell}} [(s^*_{1,N})_P \tilde{\chi}_P(\cdot)]^q,$$

or Hölder's inequality if $1 < q$ ($N > d$) to get

$$\begin{aligned}
[\sum_{|[\ell]|\preceq 2^\nu} \sum_{P\in\mathcal{Q}^{\nu,\ell}} (s^*_{1,N})_P \tilde{\chi}_P(\cdot)]^q &\leq C_d 2^{\nu(d-1)q(1-\frac{1}{q})} \sum_{|[\ell]|\preceq 2^\nu} (\sum_{P\in\mathcal{Q}^{\nu,\ell}} (s^*_{1,N})_P \tilde{\chi}_P(\cdot))^q \\
&= C_d 2^{\nu(d-1)q(1-\frac{1}{q})} \sum_{|[\ell]|\preceq 2^\nu} \sum_{P\in\mathcal{Q}^{\nu,\ell}} [(s^*_{1,N})_P \tilde{\chi}_P(\cdot)]^q,
\end{aligned}$$

because $\mathcal{Q}^{\nu,\ell}$ is a partition of $\mathbb{R}^d$. Let $\lambda > (d+1)\max(1, 1/q, 1/p)$,

$$\begin{aligned}
\|f\|_{\mathbf{F}_1} &\lesssim \left\| \left(\sum_{P\in\mathcal{Q}_{AB}} [|P_\nu|^{\frac{2d}{d+1}(-\frac{\alpha_1}{d}-1)+1} 2^{\nu(d-1)(1-\frac{1}{q})_+} (s^*_{1,\lambda})_P \tilde{\chi}_P(\cdot)]^q \right)^{1/q} \right\|_{L^p} \\
&\lesssim \left\| \left(\sum_{P\in\mathcal{Q}_{AB}} [|P|^{-\alpha_2} (s^*_{1,\lambda})_P \tilde{\chi}_P(\cdot)]^q \right)^{1/q} \right\|_{L^p} = \left\| \mathbf{s}^*_{1,\lambda} \right\|_{\mathbf{f}_2}.
\end{aligned}$$

By Lemma 5.4 and Theorem 5.5 the proof is complete.

■

7.5. Vanishing norms of non vanishing functions on Triebel-Lizorkin spaces.

We now show that there exist sequences of non vanishing functions in the norm of any of the shear anisotropic or isotropic spaces that vanish in the norm of the other space.

Theorem 7.8. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$, $0 < q_1, q_2 \leq \infty$ and $0 < p_1, p_2 < \infty$. Then, there exist sequences of functions $\{f^{(j)}\}_{j\geq 0}$ such that $\|f^{(j)}\|_{\mathbf{F}^{\alpha_2,q_2}_{p_2}(AB)} \approx 1$, but that $\|f^{(j)}\|_{\mathbf{F}^{\alpha_1,q_1}_{p_1}} \to 0$, $j \to \infty$, if $2(\alpha_1 + d) < (d+1)(\alpha_2 - \frac{1}{p_2} + 1) + \frac{d}{p_1}$.*

Proof. For $j \geq 0$, let $P_j = P_{j,0,0} \in \mathcal{Q}_{AB}$ and define $\mathbf{s}^{(j)} = \{s^{(j)}_Q\}_{Q\in\mathcal{Q}_{AB}}$ such that

$$s^{(j)}_Q = \left\{ \begin{array}{cc} 0 & \text{if } \ Q \neq P_j \\ |P_j|^{\alpha_2 - \frac{1}{p_2} + \frac{1}{2}} & \text{if } \ Q = P_j. \end{array} \right\}$$

Then, $\left\|\mathbf{s}^{(j)}\right\|_{\mathbf{f}_{p_2}^{\alpha_2,q_2}} = 1$, for all $j \geq 0$. Thus, $f^{(j)}(x) = \sum_{Q\in\mathcal{Q}_{AB}} s_Q^{(j)}\psi_Q(x) = |P_j|^{\alpha_2-\frac{1}{p_2}+\frac{1}{2}}$ $\psi_{j,0,0}(x) \in \mathbf{F}_{p_2}^{\alpha_2,q_2}(AB)$ with $\|f\|_{\mathbf{F}_{p_2}^{\alpha_2,q_2}(AB)} \approx 1$. From the compact support conditions on $\hat{\psi}$ and $\hat{\varphi}$, we have

$$\begin{aligned}
\left\|f^{(j)}\right\|_{\mathbf{F}_{p_1}^{\alpha_1,q_1}} &= \left\|\left(\sum_{\nu=0}^{\infty}[2^{\nu\alpha_1}\left|f^{(j)} * \varphi_{2^\nu I}\right|]^{q_1}\right)^{1/q_1}\right\|_{L^{p_1}} \\
&\lesssim \left\|2^{2j\alpha_1}\left|f^{(j)} * \varphi_{2^{2j}I}\right|\right\|_{L^{p_1}}.
\end{aligned}$$

Lemma 3.2 yields

$$\begin{aligned}
\left|f^{(j)} * \varphi_{2^{2j}I}(x)\right| &= \left|\int_{\mathbb{R}^d}|P_j|^{\alpha_2-\frac{1}{p_2}+\frac{1}{2}}|\det A|^{j/2}\,\psi(A^j(x-y))2^{2jd}\varphi(2^{2j}y)dy\right| \\
&\lesssim |P_j|^{(\alpha_2-\frac{1}{p_2})}\,2^{2jd}\int_{\mathbb{R}^d}\left|\psi(A^j(x-y))\right|\left|\varphi(2^{2j}y)\right|dy \\
&\lesssim \frac{|P_j|^{(\alpha_2-\frac{1}{p_2}+1)}\,2^{2jd}}{(1+2^j\,|x|)^N},
\end{aligned}$$

for every $N > d$. Then, for N such that $Np_1 > d$, we have

$$\begin{aligned}
\left\|f^{(j)}\right\|_{\mathbf{F}_{p_1}^{\alpha_1,q_1}} &\leq C_{N,q_1}\left(\int_{\mathbb{R}^d}2^{2j\alpha_1p_1}\cdot\frac{[|P_j|^{(\alpha_2-\frac{1}{p_2}+1)}\,2^{2jd}]^{p_1}}{(1+2^j\,|x|)^{Np_1}}dx\right)^{1/p_1} \\
&= C_{N,p_1,q_1}2^{2j\alpha_1-(d+1)j(\alpha_2-\frac{1}{p_2}+1)+2jd-\frac{dj}{p_1}},
\end{aligned}$$

which tends to 0 as $j \to \infty$ if $2(\alpha_1+d) < (d+1)(\alpha_2-\frac{1}{p_2}+1)+\frac{d}{p_1}$. ∎

Theorem 7.9. *Let $\alpha_1,\alpha_2 \in \mathbb{R}$, $0 < q_1,q_2 \leq \infty$ and $0 < p_1,p_2 < \infty$. Then, there exist sequences of functions $\{f^{(\nu)}\}_{\nu\geq 0}$ such that $\left\|f^{(\nu)}\right\|_{\mathbf{F}_{p_1}^{\alpha_1,q_1}} \approx 1$, but that $\left\|f^{(\nu)}\right\|_{\mathbf{F}_{p_2}^{\alpha_2,q_2}(AB)}$ $\to 0$, $\nu \to \infty$, if $\frac{d-1}{q_2}+(d+1)\alpha_2+\frac{2d}{p_1} < 2\alpha_1+\frac{d}{p_2}$.*

Proof. For a sequence $\mathbf{s}^{(\nu)} = \{s_{\nu,0}\}_{j\geq 0}$ such that $|s_{\nu,0}| = |Q_\nu|^{\frac{\alpha_1}{d}-\frac{1}{p_1}+\frac{1}{2}}$, $\left\|\mathbf{s}^{(\nu)}\right\|_{\mathbf{f}_{p_1}^{\alpha_1,q_1}} =$ 1, for all $\nu \geq 0$. This means that $f^{(\nu)}(x) = s_{\nu,0}\varphi_{\nu,0}(x) \in \mathbf{F}_{p_1}^{\alpha_1,q_1}$ and $\left\|f^{(\nu)}\right\|_{\mathbf{F}_{p_1}^{\alpha_1,q_1}} \approx 1$. Consider the subsequence $f^{(2j)}$. The conditions on the compact support of $\hat{\varphi}$ and $\hat{\psi}$ give

$$\left\|f^{(2j)}\right\|_{\mathbf{F}_2} \lesssim \left\|\left(\sum_{|[\ell]|\preceq 2^j}[|P_j|^{-\alpha_2}\,|Q_{2j}|^{\frac{\alpha_1}{d}-\frac{1}{p_1}+\frac{1}{2}}\,|\varphi_{2j,0} * \psi_{A^{-j}B^{-\ell}}|]^{q_2}\right)^{1/q_2}\right\|_{L^{p_2}}.$$

Lemma 3.2 yields

$$\begin{aligned}
|\varphi_{2j,0} * \psi_{A^{-j}B^{-\ell}}(x)| &= \left|\int_{\mathbb{R}^2}2^{2jd/2}\varphi(2^{2j}y)\,|\det A|^j\,\psi(B^\ell A^j(x-y))dy\right| \\
&\lesssim |Q_{2j}|^{-\frac{1}{2}}\,|P_j|^{-1}\int_{\mathbb{R}^2}\left|\psi(B^\ell A^j(x-y))\right|\left|\varphi(2^{2j}y)\right|dy \\
&\lesssim \frac{|Q_{2j}|^{-\frac{1}{2}}}{(1+2^j\,|x|)^N},
\end{aligned}$$

for every $N>d$ and all ℓ such that $|[\ell]|\preceq 2^j$. Then, since $|\{\ell:|[\ell]|\preceq 2^j\}|\lesssim c_d 2^{j(d-1)}$ for $N>d$ such that $Np_2>d$,

$$\begin{aligned}\left\|f^{(2j)}\right\|_{\mathbf{F}_2} &\leq 2^{\frac{j(d-1)}{q_2}}\left|P_j\right|^{-\alpha_2}\left|Q_{2j}\right|^{\frac{\alpha_1}{d}-\frac{1}{p_1}}\left(\int_{\mathbb{R}^2}\frac{dx}{(1+2^j|x|)^{Np_2}}\right)^{1/p_2}\\ &\lesssim 2^{\frac{j(d-1)}{q_2}}2^{j(d+1)\alpha_2-2j\alpha_1+\frac{2jd}{p_1}-\frac{jd}{p_2}},\end{aligned}$$

which tends to 0, as $j\to\infty$, if $\frac{d-1}{q_2}+(d+1)\alpha_2+\frac{2d}{p_1}<2\alpha_1+\frac{d}{p_2}$. ■

8. Proofs

8.1. **Proofs for Section 3.** In order to prove Lemma 3.1 we need two previous results.

Lemma 8.1. *Let A^j and $B^{[\ell]}$ be as in Section 1. Then,*

$$C_d 2^j|x|<\left|B^{[\ell]}A^jx\right|,$$

for every $j\geq 0$, $|[\ell]|\preceq 2^j$ and all $x\in\mathbb{R}^d$ with $C_d=2^{-d+1}$.

Proof. It suffices to prove the lemma for $x\in\partial\mathbb{U}:=\{y\in\mathbb{R}^d: y_1^2+\ldots+y_d^2=1\}$. From (2.8) and since $\|\cdot\|_{\ell^1(\mathbb{R}^{d-1})}\leq\sqrt{d-1}\,\|\cdot\|_{\ell^2(\mathbb{R}^{d-1})}$ and $|\ell_n|\leq 2^j$, $n=1,\ldots,d-1$, we have

$$\begin{aligned}\left|B^{[\ell]}A^jx\right| &\geq \left|2^{2j}x_1+2^j\ell_1x_2+\ldots+2^j\ell_{d-1}x_d\right|\\ &\geq \left|2^{2j}|x_1|-\left|2^j\ell_1x_2+\ldots+2^j\ell_{d-1}x_d\right|\right|\\ &\geq 2^{2j}\left||x_1|-\sqrt{d-1}(\sum_{n=2}^d|x_n|^2)^{1/2}\right|\\ &= 2^{2j}\left||x_1|-\sqrt{d-1}(1-|x_1|^2)^{1/2}\right|,\end{aligned}$$

because $x_1^2+\ldots+x_d^2=1$. Consider $|x_1|^2\geq(2^{2(d-1)}-1)/2^{2(d-1)}$. Then,

$$\begin{aligned}\left|B^{[\ell]}A^jx\right| &\geq 2^{2j}\left(\sqrt{\frac{2^{2(d-1)}-1}{2^{2(d-1)}}}-\sqrt{d-1}\sqrt{1-\frac{2^{2(d-1)}-1}{2^{2(d-1)}}}\right)\\ &\geq 2^{2j}\left(\frac{2^{2(d-1)}-d}{2^{2(d-1)}+2^{(d-1)}\sqrt{d}}\right)\geq 2^{2(j-1)},\end{aligned}$$

because $d\geq 2$. When $|x_1|^2<(2^{2(d-1)}-1)/2^{2(d-1)}$, $x\in\partial\mathbb{U}$ implies $|x_2|^2+\cdots+|x_d|^2>2^{-2(d-1)}$. Therefore, $\left|B^{[\ell]}A^jx\right|>2^{j-(d-1)}$.

Similarly one can prove $A^{-j}B^{-[\ell]}x>2^{-2(j-1)}|x|$, $j\geq 0$, $[\ell]\preceq 2^j$. ■

Lemma 8.2. *Let $g,h\in\mathcal{S}$. Then, for every $N>d$, $i=j-1,j,j+1\geq 0$, $|[m]|\preceq 2^i$ and $|[\ell]|\preceq 2^j$ there exist $C_N>0$ such that*

$$|g_{j,\ell,k}*h_{i,m,n}(x)|\leq\frac{C_N}{(1+2^i|x-A^{-i}B^{-m}n-A^{-j}B^{-\ell}k|)^N},$$

for all $x \in \mathbb{R}^d$.

Proof. Since $g, h \in \mathcal{S}$, there exists $C_N > 0$ such that $|g(x)|, |h(x)| \leq \frac{C_N}{(1+|x|)^N}$ for all $N \in \mathbb{N}$. Then,

$$|g_{j,\ell,k} * h_{i,m,n}(x)| \leq |\det A|^{(j+i)/2} \int_{\mathbb{R}^d} \frac{C_N}{(1+|B^\ell A^j y|)^N} \cdot \frac{C_N}{(1+|B^m A^i(x'-y)|)^N} dy$$

where $x' = x - A^{-j}B^{-\ell}k - A^{-i}B^{-m}n$. Notice that, since $i = j-1, j, j+1$, $|\det A|^{(j+i)/2} \simeq |\det A|^j \simeq |\det A|^i$. Following [24, §6], define

$$\begin{aligned} E_1 &= \{y \in \mathbb{R}^d : \left|B^m A^i(x'-y)\right| \leq 3\} \\ E_2 &= \{y \in \mathbb{R}^d : \left|B^m A^i(x'-y)\right| > 3, |y| \leq |x'|/2\} \\ E_3 &= \{y \in \mathbb{R}^d : \left|B^m A^i(x'-y)\right| > 3, |y| > |x'|/2\}. \end{aligned}$$

Lemma 8.1 yields the next three bounds. For $y \in E_1$ we have $1 + 2^i|x'| \leq 1 + C_d^{-1}|B^m A^i(x'-y)| + 2^i|y| \leq 1 + 3C_d^{-1} + 2^{j+1}|y| \leq 1 + 3C_d^{-1} + 2C_d^{-1}\left|B^\ell A^j y\right| \leq c_d(1 + \left|B^\ell A^j y\right|)$. If $y \in E_3$, $1 + 2^i|x'| \leq 1 + 2^{j+2}|y| \leq 1 + 2^2 C_d^{-1}\left|B^\ell A^j y\right| \leq c_d(1 + \left|B^\ell A^j y\right|)$. When $y \in E_2$, $2^{i-1}|x'| \leq 2^i|x'| - 2^i|y| < 2^i|x'-y|$, which implies $4|B^m A^i(x'-y)| = |B^m A^i(x'-y)| + 3|B^m A^i(x'-y)| \geq 3 + 3C_d 2^i|x'-y| \geq c_d'(1 + 2^i|x'-y|) \geq c_d(1 + 2^i|x'|)$. Thus, since $|j-i| \leq 1$,

$$\begin{aligned} |g_{j,\ell,k} * h_{i,m,n}(x)| &\lesssim \frac{C_N |\det A|^i}{(1+2^i|x'|)^N} \int_{E_1 \cup E_3} \frac{C_N}{(1+|B^m A^i(x'-y)|)^N} dy \\ &\quad + \frac{C_N |\det A|^i}{(1+2^i|x'|)^N} \int_{E_2} \frac{C_N}{(1+|B^\ell A^j y|)^N} dy \\ &\lesssim \frac{C_N}{(1+2^i|x'|)^N} \end{aligned}$$

for some $C_N > 0$ for every $N > d$, doing a change of variables to bound the integrals with a constant independent of i, j, l and m. The result follows by replacing back $x' = x - A^{-j}B^{-\ell}k - A^{-i}B^{-m}n$ in the estimate above.

■

As a corollary for Lemma 8.2 we have our first "almost orthogonality" property for the anisotropic and shear dilations for functions in $\mathcal{S}$.

Proof of Lemma 3.1. Identify $(j, \ell, 0)$ with P and (i, m, n) with Q. Write $|g_{A^{-j}B^{-\ell}} * h_{i,m,n}(x)| = \left||P|^{-1/2} g_P * h_Q\right|$. Since $|i-j| \leq 1$, $|P|^{-1/2} \sim |Q|^{-1/2}$. Then, Lemma 8.2 yields

$$\left||P|^{-1/2} g_P * h_Q\right| \leq \frac{C_N |P|^{-1/2}}{(1+2^j|x-x_Q|)^N} \lesssim \frac{C_N |Q|^{-1/2}}{(1+2^j|x-x_Q|)^N}.$$

■

We now present the proof for the second "almost orthogonality" result regarding dyadic isotropic dilated function and shear anisotropic dilated function.

Proof of Lemma 3.2. Since $\psi, \varphi \in \mathcal{S}$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|\psi(B^\ell A^j(x-y))\right| \left|\varphi(2^{2j}y)\right| dy \\
&\qquad\lesssim \int_{\mathbb{R}^d} \frac{1}{(1+|B^\ell A^j(x-y)|)^N} \frac{1}{(1+|2^{2j}y|)^N} dy.
\end{aligned}$$

Define

$$\begin{aligned}
E_1 &= \{y \in \mathbb{R}^d : |y| > \frac{|x|}{2}\} \\
E_2 &= \{y \in \mathbb{R}^d : |y| \leq \frac{|x|}{2}\}
\end{aligned}$$

If $y \in E_1$, $1 + 2^j|x| \leq 1 + 2^{j+1}|y| \leq 2(1 + 2^{2j}|y|)$. When $y \in E_2$, $\frac{1}{2}|x| < |x| - |y| \leq |x-y|$, which implies $4(1+\left|B^\ell A^j(x-y)\right|) \geq 1+4\left|B^\ell A^j(x-y)\right| \geq 1+4C_d 2^j|x-y| \geq c_d(1+2^j|x|)$, by Lemma 8.1. Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|\psi(B^\ell A^j(x-y))\right| \left|\varphi(2^{2j}y)\right| dy \\
&\qquad\lesssim \frac{1}{(1+2^j|x|)^N} \int_{E_1} \frac{1}{(1+|B^\ell A^j(x-y)|)^N} dy \\
&\qquad\quad + \frac{1}{(1+2^j|x|)^N} \int_{E_2} \frac{1}{(1+2^{2j}|y|)^N} dy \\
&\qquad\lesssim \left[\frac{2^{-(d+1)j}}{(1+2^j|x|)^N} + \frac{2^{-2dj}}{(1+2^j|x|)^N}\right] \lesssim \frac{2^{-(d+1)j}}{(1+2^j|x|)^N},
\end{aligned}$$

for all $N > d$.

■

The definitions of E_1, E_2, E_3 in Lema 3.2 allow us to have a "height" of 2^{-3j} and a decreasing of $(1+2^j|x|)^{-N}$.

The next proof regards the "almost orthogonality" in the Fourier domain.

Proof of Lemma 3.3. This is a direct consequence of the construction and dilation of the shearlets. Suppose $d = 2$. Since k and n are translation parameters they do not seem reflected in the support of $(\psi_{j,\ell,k})^\wedge$ or $(\psi_{i,m,n})^\wedge$. By construction and by (2.2) one scale j intersects with scales $j-1$ and $j+1$, only.

1) For one fixed scale j and by (2.3) there exist **2** overlaps at the same scale j: one with $(\psi_{j,\ell-1,k'})^\wedge$ and other with $(\psi_{j,\ell+1,k''})^\wedge$ for all $k', k'' \in \mathbb{Z}^2$.

2) Regarding scale $j-1$, one fixed $(\psi_{j,\ell,k})^\wedge$ overlaps with **3** other shearlets $(\psi_{j-1,m,k'})^\wedge$ at most for all $k, k' \in \mathbb{Z}^2$ because of 1) and because the supports of the shearlets at scale $j-1$ have larger width than those of scale j.

3) For a fixed scale j consider the next three regions: supp $(\psi_{j,\ell-1,k})^\wedge \cap$supp $(\psi_{j,\ell,k'})^\wedge = R_{-1}$, supp $(\psi_{j,\ell,k'})^\wedge \cap$ supp $(\psi_{j,\ell+1,k''})^\wedge = R_{+1}$ and supp $(\psi_{j,\ell,k'})^\wedge \setminus (R_{-1} \cup R_{+1}) = R_0$. Again by construction, there can only be two overlaps for each ξ at any scale. Then, there exist at most two shearlets at scale $j+1$ that overlap with each of the three regions R_i, $i = -1, 0, +1$ in scale j: an aggregate of **6** for all translation parameters $k, k', k'' \in \mathbb{Z}^2$ at any scale j or $j+1$.

Summing the number of overlaps at each scale gives the result for $d = 2$.

For the general case d apply the same argument above for every perpendicular direction to $\mathfrak{d}$.

■

Proof of Lemma 3.7. Suppose first that $g \in \mathcal{S}$. We can express $\hat{g}$ by its Fourier series as

$$\hat{g}(\xi) = \sum_{k\in\mathbb{Z}^d} |\det A|^{-j/2}\, \mathbf{e}^{-2\pi i\xi A^{-j}B^{-\ell}k} \cdot \left(\int_{QB^\ell A^j} \hat{g}(\omega) \cdot |\det A|^{-j/2}\, \mathbf{e}^{2\pi i\omega A^{-j}B^{-\ell}k} d\omega \right).$$

By the Fourier inversion formula in $\hat{\mathbb{R}}^d$ we have

$$\hat{g}(\xi) = \sum_{k\in\mathbb{Z}^2} |\det A|^{-j/2}\, \mathbf{e}^{-2\pi i\xi A^{-j}B^{-\ell}k} \cdot g(A^{-j}B^{-\ell}k), \qquad \xi \in QB^\ell A^j.$$

Since $\hat{g}$ has compact support, $g(A^{-j}B^{-\ell}k)$ makes sense (by the Paley-Wiener theorem). Since supp $\hat{h} \subset QB^\ell A^j$ and $g * h = (\hat{g}\hat{h})^\vee$,

$$\begin{aligned} g * h &= \sum_{k\in\mathbb{Z}^d} |\det A|^{-j}\, g(A^{-j}B^{-\ell}k)[\mathbf{e}^{-2\pi i\xi A^{-j}B^{-\ell}k}\hat{h}(\cdot)]^\vee \\ &= \sum_{k\in\mathbb{Z}^d} |\det A|^{-j}\, g(A^{-j}B^{-\ell}k)h(x - A^{-j}B^{-\ell}k), \end{aligned}$$

which proves the convergence for $g \in \mathcal{S}$. To remove the assumption $g \in \mathcal{S}$, we apply a standard regularization argument to a $g \in \mathcal{S}'$ as done in p. 22 of [31] or in Lemma A.4 of [16]. Let $\gamma \in \mathcal{S}$ satisfy supp $\hat{\gamma} \subset B(0,1)$, $\hat{\gamma}(\xi) \geq 0$ and $\gamma(0) = 1$. By Fourier inversion $|\gamma(x)| \leq 1$ for all $x \in \mathbb{R}^d$. For $0 < \delta < 1$, let $g_\delta(x) = g(x)\gamma(\delta x)$. Then, supp $\hat{g}_\delta$ is also compact, $g_\delta \in \mathcal{S}$, $|g_\delta| \leq |g|$ for all $x \in \mathbb{R}^d$ and $g_\delta \to g$ uniformly on compact sets as $\delta \to 0$. Applying the previous result to g_δ and letting $\delta \to 0$ we obtain the result for general $\mathcal{S}'$. This regularization argument (and, in fact, the whole proof) is the same used in Lemma(6.10) in [17].

■

Proof of Lemma 3.8. By the Paley-Wiener-Schwartz theorem g is of exponential type, slowly increasing and its point-wise values make sense (see Theorem 7.3.1 in [25]). Let $h \in \mathcal{S}$ satisfy supp $\hat{h} \subseteq [-1,1]^d$ with $\hat{h}(\xi) = 1$ for $\xi \in [-\frac{1}{2}, \frac{1}{2}]^d$. Write $g^y(x) = g(x + y)$. Then, as in the proof of Lemma 3.7

$$g(x + y) = h_{A^{-j}B^{-\ell}} * g^y(x) = \sum_{\kappa\in\mathbb{Z}^d} |\det A|^{-j}\, g(A^{-j}B^{-\ell}\kappa + y)\, |\det A|^j\, h(B^\ell A^j x - \kappa).$$

Therefore, for any $y \in Q_{j,\ell,k}$,

$$\sup_{z\in Q_{j,\ell,k}} |g(z)| \leq \sup_{|x|<\operatorname{diam} Q_{j,\ell,0}} |g(x + y)| \leq \sum_{\kappa\in\mathbb{Z}^d} \left|g(A^{-j}B^{-\ell}\kappa + y)\right| \sup_{|x|<\operatorname{diam} Q_{j,\ell,0}} \left|h(B^\ell A^j x - \kappa)\right|.$$

But $h \in \mathcal{S}$ implies that for any $M > 1$,

$$\sup_{|x|<\operatorname{diam} Q_{j,\ell,0}} \left|h(B^\ell A^j x - \kappa)\right| \leq \frac{C_M}{(1 + |\kappa|)^M}.$$

Taking M sufficiently large and applying the p-triangle inequality $|a+b|^p \leq |a|^p + |b|^p$ if $0 < p \leq 1$ or Hölder's inequality if $p > 1$, we obtain for any $y \in Q_{j,\ell,k}$,

$$\sup_{z \in Q_{j,\ell,k}} |g(z)|^p \leq C_p \sum_{\kappa \in \mathbb{Z}^d} \frac{\left|g(A^{-j}B^{-\ell}\kappa + y)\right|^p}{(1+|\kappa|)^{d+1}}.$$

Integrating with respect to y over the dyadic cube $Q_{j,k}$ yields

$$2^{-jd} \sup_{z \in Q_{j,\ell,k}} |g(z)|^p \leq C_p \sum_{\kappa \in \mathbb{Z}^d} (1+|\kappa|)^{-(d+1)} \int_{Q_{j,k}} \left|g(A^{-j}B^{-\ell}\kappa + y)\right|^p dy$$

Summing over $k \in \mathbb{Z}^d$,

$$\begin{aligned} |Q_j|^{\frac{d}{d+1}} \sum_{k \in \mathbb{Z}^d} \sup_{z \in Q_{j,\ell,k}} |g(z)|^p &\leq C_p \sum_{\kappa \in \mathbb{Z}^d} (1+|\kappa|)^{-(d+1)} \int_{\mathbb{R}^d} \left|g(A^{-j}B^{-\ell}\kappa + y)\right|^p dx \\ &= C_p \|g\|_{L^p}^p \sum_{\kappa \in \mathbb{Z}^d} (1+|\kappa|)^{-(d+1)} = C_p' \|g\|_{L^p}^p, \end{aligned}$$

which finishes the proof.

■

8.2. **Proofs for Section 5.** To prove our results we follow [16], [24, §6.3], [5] and [31, §1.3]. Some previous well known definitions and results are necessary.

Definition 8.3. *For a function g defined on $\mathbb{R}^d$ and for a real number $\lambda > 0$ the **Peetre's maximal function** (see Lemma 2.1 in* [29]*) is*

$$g_\lambda^*(x) = \sup_{y \in \mathbb{R}^d} \frac{|g(x-y)|}{(1+|y|)^{d\lambda}}, \quad x \in \mathbb{R}^d.$$

Lemma 8.4. *Let $g \in \mathcal{S}'(\mathbb{R}^d)$ be such that supp $(\hat{g}) \subseteq \{\xi \in \hat{\mathbb{R}}^d : |\xi| \leq R\}$ for some $R > 0$. Then, for any real $\lambda > 0$ there exists a $C_\lambda > 0$ such that, for $|\alpha| = 1$,*

$$(\partial^\alpha g)_\lambda^*(x) \leq C_\lambda g_\lambda^*(x), \quad x \in \mathbb{R}^d.$$

Proof. Since $g \in \mathcal{S}'$ has compact support in the Fourier domain, g is regular. More precisely, by the Paley-Wiener-Schwartz theorem g is slowly increasing (at most polinomialy) and infinitely differentiable (e.g., Theorem 7.3.1 in [25]). Let γ be a function in the Schwartz class such that $\hat{\gamma}(\xi) = 1$ if $|\xi| \leq R$. Then, $\hat{\gamma}(\xi)\hat{g}(\xi) = \hat{g}(\xi)$ for all $\xi \in \hat{\mathbb{R}}^d$. Hence, $\gamma * g = g$ and $\partial^\alpha g = \partial^\alpha \gamma * g$. Moreover,

$$\begin{aligned} |\partial^\alpha g(x-y)| &= \left|\int_{\mathbb{R}^d} \partial^\alpha \gamma(x-y-z) g(z) dz\right| = \left|\int_{\mathbb{R}^d} \partial^\alpha \gamma(w-y) g(x-w) dw\right| \\ &\leq \int_{\mathbb{R}^d} |\partial^\alpha \gamma(w-y)| (1+|w-y|)^{d\lambda} (1+|y|)^{d\lambda} \frac{|g(x-w)|}{(1+|w|)^{d\lambda}} dw, \end{aligned}$$

because of the triangular inequality. Therefore,

$$|\partial^\alpha g(x-y)| \leq g_\lambda^*(x)(1+|y|)^{d\lambda} \int_{\mathbb{R}^d} |\partial^\alpha \gamma(w-y)| (1+|w-y|)^{d\lambda} dw.$$

Since $\gamma \in \mathcal{S}$, the last integral equals a finite constant c_λ, independent of y, and we obtain

$$|\partial^\alpha g(x-y)| \le c_\lambda g_\lambda^*(x)(1+|y|)^{d\lambda},$$

which shows the desired result. ■

We have a relation between the Hardy-Littlewood maximal function $\mathcal{M}(|g|^{1/\lambda})(x)$ and the Peetre's maxmal function g_λ^*.

Lemma 8.5. *Let $\lambda > 0$ and $g \in \mathcal{S}'$ be such that supp $(\hat{g}) \subseteq \{\xi \in \hat{\mathbb{R}}^d : |\xi| \le R\}$ for some $R > 0$. Then, there exists a constant $C_\lambda > 0$ such that*

$$g_\lambda^*(x) \le C_\lambda \left(\mathcal{M}(|g|^{1/\lambda})(x)\right)^\lambda, \quad x \in \mathbb{R}^d.$$

Proof. Since g is band-limited, g is differentiable on $\mathbb{R}^d$ (by the Paley-Wiener-Schwartz theorem), so we can consider the pointwise values of g. Let $x, y \in \mathbb{R}^d$ and $0 < \delta < 1$. Choose $z \in \mathbb{R}^d$ such that $z \in B_\delta(x-y)$. We apply the mean value theorem to g and the endpoints $x-y$ and z to get

$$|g(x-y)| \le |g(z)| + \delta \sup_{w : w \in B_\delta(x-y)} (|\nabla g(w)|).$$

Taking the $(1/\lambda)^{\text{th}}$ power and integrating with respect to the variable z over $B_\delta(x-y)$, we obtain

$$\begin{aligned} |g(x-y)|^{1/\lambda} &\le \frac{c_\lambda}{|B_\delta(x-y)|} \int_{B_\delta(x-y)} |g(z)|^{1/\lambda}\, dz \\ &\quad + c_\lambda \delta^{1/\lambda} \sup_{w : w \in B_\delta(x-y)} (|\nabla g(w)|)^{1/\lambda}. \end{aligned} \tag{8.1}$$

Since $B_\delta(x-y) \subset B_{\delta+|y|}(x)$,

$$\int_{B_\delta(x-y)} |g(z)|^{1/\lambda}\, dz \le \int_{B_{\delta+|y|}(x)} |g(z)|^{1/\lambda}\, dz \le \left|B_{\delta+|y|}(x)\right| \mathcal{M}(|g|^{1/\lambda})(x),$$

and the sup term on the right hand side of (8.1) is bounded by

$$\begin{aligned} \sup_{w : w \in B_{\delta+|y|}(x)} (|\nabla g(w)|)^{1/\lambda} &= \sup_{t : |t| < \delta+|y|} (|\nabla g(x-t)|)^{1/\lambda} \\ &\lesssim (1+\delta+|y|)^d \left[(\nabla g)_\lambda^*(x)\right]^{1/\lambda}. \end{aligned}$$

Substituting these last two inequalities in (8.1) yields

$$\begin{aligned} |g(x-y)|^{1/\lambda} &\le c_\lambda \frac{\left|B_{\delta+|y|}(x)\right|}{|B_\delta(x-y)|} \mathcal{M}(|g|^{1/\lambda})(x) \\ &\quad + c_\lambda \delta^{1/\lambda}(1+\delta+|y|)^d \left[(\nabla g)_\lambda^*(x)\right]^{1/\lambda}, \end{aligned}$$

and since $\left|B_{\delta+|y|}(x)\right| / |B_\delta(x-y)| = (\delta+|y|)^d/\delta^d$, we get

$$\begin{aligned} |g(x-y)|^{1/\lambda} &\le c_\lambda \frac{(\delta+|y|)^d}{\delta^d} \mathcal{M}(|g|^{1/\lambda})(x) \\ &\quad + c_\lambda \delta^{1/\lambda}(1+\delta+|y|)^d \left[(\nabla g)_\lambda^*(x)\right]^{1/\lambda}. \end{aligned}$$

Taking the λ^{th} power yields

$$\frac{|g(x-y)|}{(1+|y|)^{d\lambda}} \le c'_\lambda \left\{ \frac{1}{\delta^{d\lambda}} [\mathcal{M}(|g|^{1/\lambda})(x)]^\lambda + \delta \left[(\nabla g)^*_\lambda(x)\right] \right\},$$

since $\delta < 1$ implies $(1+\delta+|y|) \le 2(1+|y|)$. Taking δ small enough so that $c'_\lambda C_\lambda \delta < 1/(2d)$ (where C_λ is the constant in Lemma 8.4) we obtain

$$g^*_\lambda(x) \le c_\lambda [\mathcal{M}(|g|^{1/\lambda})(x)]^\lambda + \frac{1}{2} g^*_\lambda(x).$$

Assume for the moment that $g \in \mathcal{S}$, hence $g^*_\lambda(x) < \infty$. So, we can subtract the second term in the right-hand side of the previous inequality from the left-hand side of the previous inequality and complete the proof for $g \in \mathcal{S}$. To remove this assumption one uses the same standard regularization argument as in the proof of Lemma 3.7. ∎

Lemmata 8.4 and 8.5 are Peetre's inequality for $f \in \mathcal{S}'$ whose proofs can be found in the references above and we reproduce them for completeness.

Proof of Lemma 5.3. Let $g(x) = (\psi_{A^{-j}B^{-\ell}} * f)(x)$. Since ψ is band-limited, so is g. On one hand, $j \ge 0$ implies $C_d 2^j |y| \le \left|B^\ell A^j y\right|$, by Lemma 8.1. Thus,

$$\begin{aligned}
g^*_\lambda(t) &= \sup_{y\in\mathbb{R}^d} \frac{|g(t-y)|}{(1+|y|)^{d\lambda}} \ge \sup_{y\in\mathbb{R}^d} \frac{|(\psi_{A^{-j}B^{-\ell}} * f)(t-y)|}{(1+2^j|y|)^{d\lambda}} \\
&= \sup_{y\in\mathbb{R}^d} \frac{|(\psi_{A^{-j}B^{-\ell}} * f)(t-y)|}{2^{d\lambda}(2^{-1}+2^{j-1}|y|)^{d\lambda}} \\
&\ge C_{d,\lambda} \sup_{y\in\mathbb{R}^d} \frac{|(\psi_{A^{-j}B^{-\ell}} * f)(t-y)|}{(1+|B^\ell A^j y|)^{d\lambda}} = C_{d,\lambda} \left|(\psi^{**}_{j,\ell,\lambda})(t)\right|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathcal{M}(|g|^{1/\lambda})(t) &= \sup_{r>0} \frac{1}{|B_r(t)|} \int_{B_r(t)} |(\psi_{A^{-j}B^{-\ell}} * f)(y)|^{1/\lambda}\, dy \\
&= \mathcal{M}(|(\psi_{A^{-j}B^{-\ell}} * f|^{1/\lambda})(t).
\end{aligned}$$

The result follows from Lemma 8.5 with $t = x$. ∎

To prove Lemma 5.4 we need the next result.

Lemma 8.6. *Let $i \ge j \ge 0$ and $0 < a \le r$. Also, let Q and P be identified with (i,m,n) and (j,ℓ,k), respectively. Then, for all $N > (d+1)r/a$, any sequence $\{s_P\}_{P\in\mathcal{Q}^{j,\ell}}$ of complex numbers and any $x \in Q$,*

$$(s^*_{r,N})_Q := \left(\sum_{P\in\mathcal{Q}^{j,\ell}} \frac{|s_P|^r}{(1+2^j|x_Q - x_P|)^N} \right)^{1/r} \le C_{a,r,d} \left[\mathcal{M}\left(\sum_{P\in\mathcal{Q}^{j,\ell}} |s_P|^a \chi_P \right)(x) \right]^{1/a}.$$

Moreover, when $i = j$,

$$\sum_{P\in\mathcal{Q}^{j,\ell}} \left[(s^*_{r,N})_P \tilde{\chi}_P(x)\right]^q \le C_{a,r,d} \left[\mathcal{M}\left(\sum_{P\in\mathcal{Q}^{j,\ell}} (|s_P|\,\tilde{\chi}_P)^a \right)(x) \right]^{q/a}.$$

Proof. Identify (i,m,n) and (j,ℓ,k) with Q and P, respectively. Then, $x_Q = A^{-i}B^{-m}n$ and $x_P = A^{-j}B^{-\ell}k$. Let $\mathcal{Q}^{j,\ell} := \{Q_{j,\ell,k} : k \in \mathbb{Z}^d\}$, then $\mathcal{Q}^{j,\ell}$ is a partition of $\mathbb{R}^d$. Write $l_P = |x_Q - x_P|$. Thus, we bound the sum in the definition of $(s^*_{r,N})_Q$ as

$$\sum_{P\in\mathcal{Q}^{j,\ell}} \frac{|s_P|^r}{(1+2^j|x_Q-x_P|)^N} \le \left(\sum_{P\in\mathcal{Q}^{j,\ell}:l_P\le 1} + \sum_{P\in\mathcal{Q}^{j,\ell}:l_P>1}\right)\frac{|s_P|^r}{(1+2^j l_P)^N}.$$

Choose λ such that $N > (d+1)\lambda/d > (d+1)r/a$. Then, the inequality $(2^j l_P)^N > (2^{(d+1)j/d} l_P)^\lambda$ holds whenever $l_P > 2^{j((d+1)\lambda/d - N)/(N-\lambda)}$. So, the previous inequality is bounded by

$$\sum_{P\in\mathcal{Q}^{j,\ell}:l_P\le 1} \frac{|s_P|^r}{(1+2^j l_P)^N} + \sum_{P\in\mathcal{Q}^{j,\ell}:l_P>1} \frac{|s_P|^r}{(2^{(d+1)j/d} l_P)^\lambda}.$$

Defining

$$\begin{aligned} D_0 &= \{k\in\mathbb{Z}^d : \left|A^{-i}B^{-m}n - A^{-j}B^{-\ell}k\right| \le 1\} \\ &= \{P\in\mathcal{Q}^{j,\ell} : l_P = |x_Q - x_P| \le 1\} \end{aligned}$$

and

$$\begin{aligned} D_\nu &= \{k\in\mathbb{Z}^d : 2^{\nu-1} < 2^{(d+1)j/d}\left|A^{-i}B^{-m}n - A^{-j}B^{-\ell}k\right| \le 2^\nu\} \\ &= \{P\in\mathcal{Q}^{j,\ell} : 2^{\nu-1} < 2^{(d+1)j/d} l_P \le 2^\nu\}, \quad \nu = 1,2,3,\ldots, \end{aligned}$$

we have that

$$\begin{aligned} \sum_{P\in\mathcal{Q}^{j,\ell}} \frac{|s_P|^r}{(1+2^j l_P)^N} &\le \sum_{P\in D_0} |s_P|^r + 2^\lambda \sum_{\nu=1}^\infty \sum_{P\in D_\nu} \frac{|s_P|^r}{2^{\nu\lambda}} \\ &\le 2^\lambda \sum_{\nu=0}^\infty \sum_{P\in D_\nu} \frac{|s_P|^r}{2^{\nu\lambda}} \le 2^\lambda \sum_{\nu=0}^\infty 2^{-\nu\lambda}\left(\sum_{P\in D_\nu}|s_P|^a\right)^{r/a}, \end{aligned}$$

because $1 + 2^j l_P \ge 1$ and $a \le r$. Now, when $x \in Q_{i,m,n} = Q$ and $P \in D_\nu$ then, by the definition of D_ν, $P = Q_{j,\ell,k} \subset B_{(d+1)2^{\nu-(d+1)j/d}}(x)$ (this holds because for $j \ge 0$ the diameter of any $P = Q_{j,\ell,k}$ is less than $(d+1)$ and the intervals in the definition of the D_ν's are dyadic with $\nu \ge 0$). Thus,

$$\sum_{P\in D_\nu} |s_P|^a = |P|^{-1}\int_{B_{(d+1)2^{\nu-(d+1)j/d}}(x)} \sum_{P\in D_\nu} |s_P|^a \chi_P(y)dy.$$

Hence, writing $\left|\tilde{B}\right| = \left|B_{(d+1)2^{\nu-(d+1)j/d}}(x)\right| = C_d(d+1)^d 2^{d\nu-(d+1)j}$ we have that for $x \in Q_{i,m,n} = Q$,

$$\begin{aligned} \sum_{P\in\mathcal{Q}^{j,\ell}} \frac{|s_P|^r}{(1+2^j l_P)^N} &\le 2^\lambda \sum_{\nu=0}^\infty 2^{-\nu\lambda}\left(\frac{|P|^{-1}\left|\tilde{B}\right|}{\left|\tilde{B}\right|}\int_{\tilde{B}} \sum_{P\in D_\nu}|s_P|^a\chi_P(y)dy\right)^{r/a} \\ &\le C_{a,r,d}\sum_{\nu=0}^\infty 2^{-\nu\lambda}2^{d\nu r/a}\left(\mathcal{M}\left(\sum_{P\in D_\nu}|s_P|^a\chi_P\right)(x)\right)^{r/a} \end{aligned}$$

$$\leq C_{a,r,d}\left(\mathcal{M}\left(\sum_{P\in\mathcal{Q}^{j,\ell}}|s_P|^a\chi_P\right)(x)\right)^{r/a},$$

because $|P|^{-1}=2^{(d+1)j}$ and $\lambda>dr/a$.

To prove the second inequality multiply both sides by $\tilde{\chi}_Q(x)$, rise to the power q and sum over $Q\in\mathcal{Q}^{j,\ell}$ to get

$$\begin{aligned}\sum_{Q\in\mathcal{Q}^{j,\ell}}\left[(s^*_{r,N})_Q\tilde{\chi}_Q(x)\right]^q &\leq C\sum_{Q\in\mathcal{Q}^{j,\ell}}\left[\mathcal{M}\left(\sum_{P\in\mathcal{Q}^{j,\ell}}|s_P|^a\chi_P\right)(x)\right]^{q/a}\tilde{\chi}^q_Q(x)\\ &= C\sum_{Q\in\mathcal{Q}^{j,\ell}}\left[\mathcal{M}\left(\sum_{P\in\mathcal{Q}^{j,\ell}}(|s_P|\,\tilde{\chi}_P)^a\right)(x)\right]^{q/a}\chi_Q(x)\\ &= C\left[\mathcal{M}\left(\sum_{P\in\mathcal{Q}^{j,\ell}}(|s_P|\,\tilde{\chi}_P)^a\right)(x)\right]^{q/a},\end{aligned}$$

since $\mathcal{Q}^{j,\ell}$ is a partition of $\mathbb{R}^d$.

■

Proof of Lemma 5.4. Let λ be such that $N>(d+1)\lambda/d>(d+1)\max(1,r/q,r/p)$. If $r<\min(q,p)$, choose $a=r$. Otherwise, if $r\geq\min(q,p)$, choose a such that $r/(\lambda/d)<a<\min(r,q,p)$. It is always possible to choose such an a since $\lambda/d>\max(1,r/q,r/p)$ implies $r/(\lambda/d)<\min(r,q,p)$. In both cases we have that

$$0<a\leq r<\infty,\quad \lambda>dr/a,\quad q/a>1,\quad p/a>1.$$

The previous argument is similar to that in [5]. Then, by Lemma 8.6 and Theorem 3.10

$$\begin{aligned}\left\|\mathbf{s}^*_{r,N}\right\|_{\mathbf{f}^{\alpha,q}_p(AB)} &= \left\|\left(\sum_{P\in\mathcal{Q}_{AB}}(|P|^{-\alpha}(s^*_{r,N})_P\tilde{\chi}_P)^q\right)^{1/q}\right\|_{L^p}\\ &= \left\|\left(\sum_{j\geq 0}\sum_{|[\ell]|\preceq 2^j}|Q_{j,\ell}|^{-\alpha q}\sum_{P\in\mathcal{Q}^{j,\ell}}(s^*_{r,N})^q_P\tilde{\chi}^q_P\right)^{1/q}\right\|_{L^p}\\ &\leq C_{a,r,d}\left\|\left(\sum_{j\geq 0}\sum_{|[\ell]|\preceq 2^j}|Q_{j,\ell}|^{-\alpha q}\left[\mathcal{M}\left(\sum_{P\in\mathcal{Q}^{j,\ell}}(|s_P|\,\tilde{\chi}_P)^a\right)\right]^{q/a}\right)^{1/q}\right\|_{L^p}\\ &= C_{a,r,d}\left\|\left(\sum_{j\geq 0}\sum_{|[\ell]|\preceq 2^j}\left[\mathcal{M}\left(\sum_{P\in\mathcal{Q}^{j,\ell}}(|P|^{-\alpha}|s_P|\,\tilde{\chi}_P)^a\right)\right]^{q/a}\right)^{1/q}\right\|_{L^p}\end{aligned}$$

$$
\begin{aligned}
&= C_{a,r,d} \left\| \left(\sum_{j\geq 0} \sum_{|[\ell]|\preceq 2^j} \left[\mathcal{M} \left(\sum_{P\in\mathcal{Q}^{j,\ell}} (|P|^{-\alpha} |s_P| \tilde{\chi}_P)^a \right) \right]^{q/a} \right)^{a/q} \right\|_{L^{p/a}}^{1/a} \\
&\leq C_{a,r,d} \left\| \left(\sum_{j\geq 0} \sum_{|[\ell]|\preceq 2^j} \left(\sum_{P\in\mathcal{Q}^{j,\ell}} (|P|^{-\alpha} |s_P| \tilde{\chi}_P)^a \right)^{q/a} \right)^{a/q} \right\|_{L^{p/a}}^{1/a} \\
&= C_{a,r,d} \left\| \left(\sum_{j\geq 0} \sum_{|[\ell]|\preceq 2^j} \sum_{P\in\mathcal{Q}^{j,\ell}} (|P|^{-\alpha} |s_P| \tilde{\chi}_P)^q \right)^{a/q} \right\|_{L^{p/a}}^{1/a} \\
&= C_{a,r,d} \left\| \left(\sum_{j\geq 0} \sum_{|[\ell]|\preceq 2^j} \sum_{P\in\mathcal{Q}^{j,\ell}} (|P|^{-\alpha} |s_P| \tilde{\chi}_P)^q \right)^{1/q} \right\|_{L^{p}} \\
&= C_{a,r,d} \left\| \mathbf{s} \right\|_{f_p^{\alpha,q}(AB)},
\end{aligned}
$$

because $\mathcal{Q}^{j,\ell}$ is a partition of $\mathbb{R}^d$.

The reverse inequality is trivial since $|s_Q| \leq (s^*_{r,N})_Q$ always holds. ■

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Daniel Vera, Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain

E-mail address: `daniel.vera@uam.es`